# Chapter 4
# The Odd Edge Characterization and its Role in the Combinatorial Topology of the n-Dimensional Orthogonal Pseudo-Polytopes

In this chapter we will introduce some new concepts and frameworks and some definitions previously commented will be redefined. Furthermore, this chapter is related directly with Local Analysis over the nD-OPP's. The results obtained eventually will lead us to the formalization of the Extreme Vertices Model in the n-Dimensional Space (nD-EVM) in **Chapter 5**.

Manifold edges in the 1D, 2D and 3D-OPP's share the characteristic that they have an odd number of incident segments, rectangles and boxes respectively [Aguilera98]. Moreover, Extreme edges in the 4D-OPP's have also an odd number of incident 4D hyper-boxes [Pérez-Aguila03b]. As a natural extension based in the previous observations, in **Chapter 3** we defined an Odd Edge, in a combination of nD hyper-boxes, as an edge with an odd number of incident hyper-boxes. We will analyze how this very simple concept provides us important information about the combinatorial nature of the nD-OPP's from a topological point of view. The properties we obtain will perform an important and essential role when we define the foundations behind the nD-EVM.

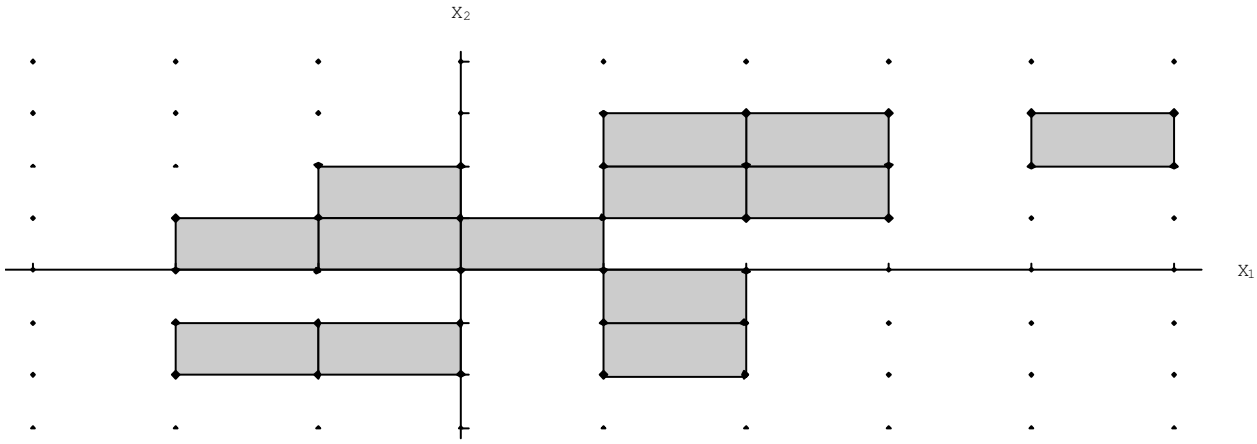



**Figure 4.1.** A 2D-OPP described by a set of rectangles.

Consider the following 2D-OPP defined by the union of the 13 rectangles shown in **Figure 4.1**. Each vertex of that 2D-OPP defines around it a combination with 1 to 4 boxes which are incident to it. It implies that our 2D-OPP can be analyzed from a local point of view by considering the geometrical and topological properties around one of its vertices. This analysis is performed only in the context of the combination of rectangles that describe one given vertex. In the following sections we will analyze nD-OPP's from a local point of view. That is, we will analyze vertices and its surrounding nD hyper-boxes.

In the other hand, a key concept to consider in this chapter is the *k-chain*. This useful tool will provide us the advantage to define equivalence relations in order to classify the (n-1)D cells in the boundary of a combination of hyper-boxes. Our objective is to define in a precise manner and in unambiguously way which (n-1)D cells are we considering in our analysis. The properties we identify through such analysis will be applied in **Chapter 5** where the nD-EVM will be defined.

## 4.1. Local Analysis Over the nD-OPP's

In **Chapter 3**, we gave a small introduction to k-chains in order to define equivalence relation $R_H$. We established that a dimensionally homogeneous complex containing only kD hyper-boxes is called a k-chain, $1 \leq k \leq n$. Moreover, a chain's boundary consists of those (k-1)D hyper-boxes that are incident to an odd number of kD hyper-boxes. If every (k-1)D hyper-box is shared by an even number of kD hyper-boxes, the chain has no boundary [Henle94]. Moreover, we introduced that the sum of two chains consists of those hyper-boxes appearing in either chain but not in both [Coxeter63]. In this section we reconsider some theory related to k-chains in order to determine in a formal way some properties related to the topology and geometry, from a local point of view, of the vertices and their incident kD hyper-boxes in an nD-OPP. Many authors aboard the study of k-chains with some slight differences according to the applications. The following examples show some important applications of k-chains:

- In [Coxeter63] is described how Poincaré applied k-chains in order to give the first correct formal proof of Euler Characteristic for n-Dimensional Polytopes[1].
- Henle [Henle94] describes the way Jordan's Theorem is generalized to n-Dimensional Space.
- Sobczyk [Sobczyk99] shows the application of Clifford Algebras to Simplicial Calculus via k-chains.
- Matveev & Polyak [Matveev02] expose finite type invariants of knots and homology 3-spheres from the "k-chains" point of view.
- In [Hocking88], [Naber00], [Agoston05], [Crossley05], among others, is shown how an important property of k-chains, which we will describe in following sections, provides the foundation of an important field in topology: Homology Theory.
- In texts such as [Spivak65], [Mikusinski02] or [Agoston05] is described the relation between differential forms and integration through the application of k-chains. This application leads to important results such as the generalization of Stokes Theorem to n-Dimensional Space. This generalization is sometimes called the *Fundamental Theorem of Calculus in Higher Dimensions* or *Generalized Fundamental Theorem of Calculus*.

We will aboard our study by considering the application of k-chains according to the definitions given in [Spivak65] and [Agoston05]. In first place, **Section 4.2** will describe some basic concepts. In **Section 4.3**, in some cases, we will give new definitions to some concepts previously discussed in **Chapters 2** and **3** in order to connect them with the language of k-chains.

## 4.2. Spivak's k-chains Fundamental Concepts

The definitions presented in this section are related mainly with [Spivak65] and [Agoston05].

**Definition 4.1:** *A <u>Singular n-Dimensional Hyper-Box</u> in $\mathbb{R}^n$ is the continuous function*

$$\begin{array}{rccc} I^n: & [0,1]^n & \rightarrow & [0,1]^n \\ & x & \sim & I^n(x) = x \end{array}$$

**Definition 4.2:** *A <u>General Singular k-Dimensional Hyper-Box</u> in the closed set $A \subset \mathbb{R}^n$ is the continuous function*

$$c:[0,1]^k \rightarrow A$$

**Definition 4.3:** *For all i, $1 \leq i \leq n$, the two singular (n-1)D hyper-boxes $\underline{I^n_{(i,0)}}$ and $\underline{I^n_{(i,1)}}$ are defined as follows: If $x \in [0,1]^{n-1}$ then*

- $I^n_{(i,0)}(x) = I^n(x_1,...,x_{i-1},0,x_i,...,x_{n-1}) = (x_1,...,x_{i-1},0,x_i,...,x_{n-1})$
- $I^n_{(i,1)}(x) = I^n(x_1,...,x_{i-1},1,x_i,...,x_{n-1}) = (x_1,...,x_{i-1},1,x_i,...,x_{n-1})$

For example, consider the singular 2D hyper-box in $\mathbb{R}^2$ defined ad

$$\begin{array}{rccc} I^2: & [0,1]^2 & \rightarrow & [0,1]^2 \\ & (x_1,x_2) & \sim & I^2(x_1,x_2) = (x_1,x_2) \end{array}$$

[1] In [Sommerville58] is presented a proof of Euler Characteristic which is developed using induction over the number of dimensions. Authors such as Coxeter, Grümbaum or McMullen & Schulte have criticized Sommerville's proof because it does not provide specific details about the way vertices, edges, …, and (n-1)D cells are connected in the polytopes' boundary ([Coxeter63], [Grümbaum03] & [McMullen02]).

If $x \in [0,1]$ then

- $I^2_{(1,0)}(x_1, x_2) = I^2(0, x_1) = (0, x_1)$
- $I^2_{(1,1)}(x_1, x_2) = I^2(1, x_1) = (1, x_1)$
- $I^2_{(2,0)}(x_1, x_2) = I^2(x_1, 0) = (x_1, 0)$
- $I^2_{(2,1)}(x_1, x_2) = I^2(x_1, 1) = (x_1, 1)$

Such functions correspond to the set of edges of the unit square in the 2D space (see **Figure 4.2**).

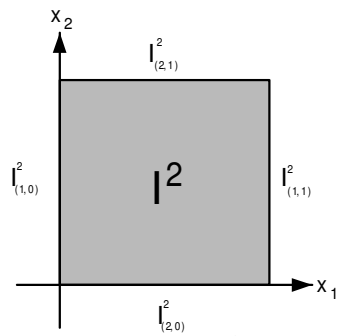



**Figure 4.2.** A unit 2D square $I^2$ and its 1D cells.

**Definition 4.4:** *In a general singular nD hyper-box c we define the* <u>*(i,α)-cell as*</u>

$$c_{(i,\alpha)} = c \circ I^n_{(i,\alpha)}$$

Let $A = \{(x_1, x_2) : -1 \le x_1 \le 0, -1 \le x_2 \le 0\}$. Let a general singular 2D hyper-box in $\mathbb{R}^2$ be defined as

$$\begin{array}{llll} c: & [0,1]^2 & \to & A \\ & (x_1, x_2) & \sim & c(x_1, x_2) = (x_1 - 1, x_2 - 1) \end{array}$$

If $x \in [0,1]$ then

- $c(I^2_{(1,0)}(x_1, x_2)) = c(0, x_1) = (-1, x_1 - 1)$
- $c(I^2_{(1,1)}(x_1, x_2)) = c(1, x_1) = (0, x_1 - 1)$
- $c(I^2_{(2,0)}(x_1, x_2)) = c(x_1, 0) = (x_1 - 1, -1)$
- $c(I^2_{(2,1)}(x_1, x_2)) = c(x_1, 1) = (x_1 - 1, 0)$

Such functions correspond to the edges of a square located in the third quadrant of 2D space.

For each general singular kD hyper-box c we will define the boundary of c (moreover, we will define the boundary of a k-chain). But, before going any further, Spivak considers more appropriate to define the boundary, for example, of $I^2$ not as the sum of four singular 1-cubes (edges) arranged counterclockwise around $I^2$ (see **Figure 4.3.a**). Instead, the following definitions will indicate in a precise way what we will consider as the orientation of an (n-1)D cell.

**Definition 4.5:** *The* <u>*orientation*</u> *of an (n-1)D cell* $c \circ I^n_{(i,\alpha)}$ *is given by* $(-1)^{\alpha+i}$.

**Definition 4.6:** *An* <u>*(n-1)D oriented cell*</u> *is given by the scalar-function product:*

$$(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}$$

For example, the orientations of the edges in a 2D unit cube, according to Spivak considerations, are given by (See **Figure 4.3.b**):

- $I^2_{(1,0)}(x_1,x_2) = I^2(0,x_1) = (0,x_1) \Rightarrow (-1)^{\alpha+i} = (-1)^{1+0} = -1$
- $I^2_{(1,1)}(x_1,x_2) = I^2(1,x_1) = (1,x_1) \Rightarrow (-1)^{\alpha+i} = (-1)^{1+1} = 1$
- $I^2_{(2,0)}(x_1,x_2) = I^2(x_1,0) = (x_1,0) \Rightarrow (-1)^{\alpha+i} = (-1)^{2+0} = 1$
- $I^2_{(2,1)}(x_1,x_2) = I^2(x_1,1) = (x_1,1) \Rightarrow (-1)^{\alpha+i} = (-1)^{2+1} = -1$

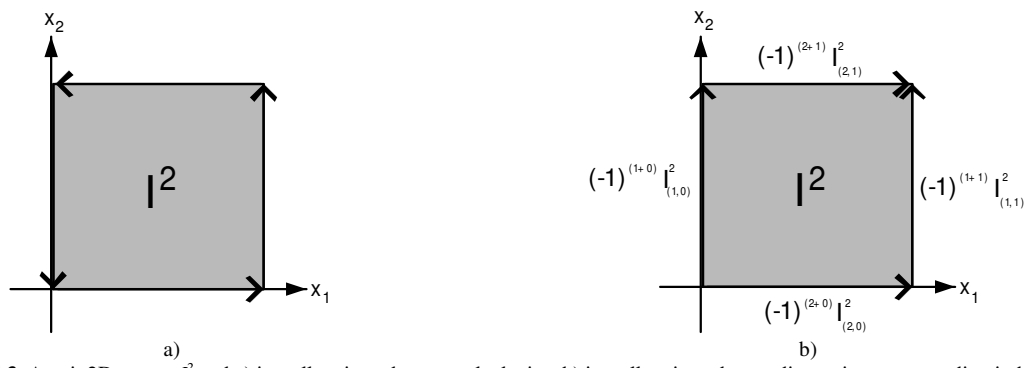



**Figure 4.3.** A unit 2D square $I^2$ and a) its cells oriented counterclockwise; b) its cells oriented according to its corresponding indexes.

**Definition 4.7:** *A formal linear combination of singular general kD hyper-boxes, $1 \leq k \leq n$, for a closed set A is called a* <u>*k-chain*</u>*.*

For example, consider the combination of boxes composed by the following general singular 3D cubes (See **Figure 4.4**):

- $c_1: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, -1 \leq x_3 \leq 0\}$
  $x \ \sim \ c_1(x) = (x_1, x_2, x_3 - 1)$
- $c_2: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3) \in \mathbb{R}^3 : -1 \leq x_1 \leq 0, 0 \leq x_2 \leq 1, -1 \leq x_3 \leq 0\}$
  $x \ \sim \ c_2(x) = (x_1 - 1, x_2, x_3 - 1)$
- $c_3: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3) \in \mathbb{R}^3 : -1 \leq x_1 \leq 0, -1 \leq x_2 \leq 0, -1 \leq x_3 \leq 0\}$
  $x \ \sim \ c_3(x) = (x_1 - 1, x_2 - 1, x_3 - 1)$
- $c_4: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3) \in \mathbb{R}^3 : -1 \leq x_1 \leq 0, -1 \leq x_2 \leq 0, 0 \leq x_3 \leq 1\}$
  $x \ \sim \ c_4(x) = (x_1 - 1, x_2 - 1, x_3)$

Through these 3D boxes we can define the following 3-chain:

$$C_1(x) + C_2(x) + C_3(x) + C_4(x) = (x_1, x_2, x_3 - 1) + (x_1 - 1, x_2, x_3 - 1) + (x_1 - 1, x_2 - 1, x_3 - 1) + (x_1 - 1, x_2 - 1, x_3)$$

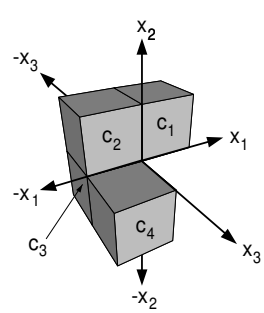



**Figure 4.4.** A combination of 3D boxes.

**Definition 4.8:** *Given a singular nD hyper-box $I^n$ we define the (n-1)-chain, called the* <u>*boundary of $I^n$*</u>*, by*

$$\partial(I^n) = \sum_{i=1}^{n} \left( \sum_{\alpha=0,1} (-1)^{i+\alpha} \cdot I^n_{(i,\alpha)} \right)$$

**Definition 4.9:** *Given a singular general nD hyper-box c we define the (n-1)-chain, <u>called the boundary of c</u>, by*

$$\partial(c) = \sum_{i=1}^{n}\left(\sum_{\alpha=0,1}(-1)^{i+\alpha}\cdot c\circ I^{n}_{(i,\alpha)}\right)$$

**Definition 4.10:** *The <u>boundary of an n-chain</u>* $\sum c_i$ *, where each* $c_i$ *is a singular general nD hyper-box, is given by*

$$\partial\left(\sum c\right) = \sum \partial(c)$$

At this point it is important to note that Spivak's notion of orientation differs slightly from the definitions given in [Coxeter63] or [Henle94], for example. The difference arises from the fact that adding k-chains, according to Coxeter or Henle, is a procedure based in addition modulo 2. However, due to -1 ≡ 1 mod 2 then each coefficient of a Spivak's k-chain can be rewritten as a coefficient in {0, 1}. We will preserve Spivak's notation.

We consider again the combination of 3D cubes, shown in **Figure 4.4**, whose corresponding 3-chain is:

$C_1(x) + C_2(x) + C_3(x) + C_4(x) = (x_1, x_2, x_3 - 1) + (x_1 - 1, x_2, x_3 - 1) + (x_1 - 1, x_2 - 1, x_3 - 1) + (x_1 - 1, x_2 - 1, x_3)$

We proceed to compute its corresponding boundary, i.e., $\partial(C_1(x) + C_2(x) + C_3(x) + C_4(x))$. It is important to observe that the pairs of boxes $C_1$ and $C_2$, $C_2$ and $C_3$; and $C_3$ and $C_4$ have common faces, i.e., there exists a face adjacency between them (see **Figure 4.4**). We will see in our computing how these faces are not present in the boundary of the combination. By applying **Definitions 4.9** and **4.10** we have:

$$\partial(c_1(x)+c_2(x)+c_3(x)+c_4(x)) = \partial(c_1(x))+\partial(c_2(x))+\partial(c_3(x))+\partial(c_4(x))$$

$$= \sum_{i=1}^{3}\left(\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\cdot c_1(I^{3}_{(i,\alpha)}(x))\right)+\sum_{i=1}^{3}\left(\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\cdot c_2(I^{3}_{(i,\alpha)}(x))\right)+\sum_{i=1}^{3}\left(\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\cdot c_3(I^{3}_{(i,\alpha)}(x))\right)+\sum_{i=1}^{3}\left(\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\cdot c_4(I^{3}_{(i,\alpha)}(x))\right)$$

At this point we must compute the oriented 2D cells of each one of the boxes, which gives (we indicate the functions that correspond to the shared faces, between two cubes, in the combination):

$$= \left[\underbrace{\boxed{-1\cdot(0,x_1,x_2-1)}+1\cdot(1,x_1,x_2-1)+1\cdot(x_1,0,x_2-1)-1\cdot(x_1,1,x_2-1)-1\cdot(x_1,x_2,-1)+1\cdot(x_1,x_2,0)}_{c_1\text{'s } 2D \text{ cells}}\right]+$$

$$\left[\underbrace{-1\cdot(-1,x_1,x_2-1)\boxed{+1\cdot(0,x_1,x_2-1)}\boxed{+1\cdot(x_1-1,0,x_2-1)}-1\cdot(x_1-1,1,x_2-1)-1\cdot(x_1-1,x_2,-1)+1\cdot(x_1-1,x_2,0)}_{c_2\text{'s } 2D \text{ cells}}\right]+$$

$$\left[\underbrace{-1\cdot(-1,x_1-1,x_2-1)+1\cdot(0,x_1-1,x_2-1)+1\cdot(x_1-1,-1,x_2-1)\boxed{-1\cdot(x_1-1,0,x_2-1)}-1\cdot(x_1-1,x_2-1,-1)\boxed{+1\cdot(x_1-1,x_2-1,0)}}_{c_3\text{'s } 2D \text{ cells}}\right]+$$

$$\left[\underbrace{-1\cdot(-1,x_1-1,x_2)+1\cdot(0,x_1-1,x_2)+1\cdot(x_1-1,-1,x_2)-1\cdot(x_1-1,0,x_2)\boxed{-1\cdot(x_1-1,x_2-1,0)}+1\cdot(x_1-1,x_2-1,1)}_{c_4\text{'s } 2D \text{ cells}}\right]$$

The shared faces have the same function but with opposite orientation, and therefore they will cancel out from the chain:

$$=\left[\underbrace{-1\cdot(0,x_1,x_2-1)+1\cdot(0,x_1,x_2-1)}_{\text{Shared faces between } c_1 \text{ and } c_2}\right]+\left[\underbrace{1\cdot(x_1-1,0,x_2-1)-1\cdot(x_1-1,0,x_2-1)}_{\text{Shared faces between } c_2 \text{ and } c_3}\right]+\left[\underbrace{1\cdot(x_1-1,x_2-1,0)-1\cdot(x_1-1,x_2-1,0)}_{\text{Shared faces between } c_3 \text{ and } c_4}\right]+$$

$$1\cdot(1,x_1,x_2-1)+1\cdot(x_1,0,x_2-1)-1\cdot(x_1,1,x_2-1)-1\cdot(x_1,x_2,-1)+1\cdot(x_1,x_2,0)-1\cdot(-1,x_1,x_2-1)-1\cdot(x_1-1,1,x_2-1)$$
$$-1\cdot(x_1-1,x_2,-1)+1\cdot(x_1-1,x_2,0)-1\cdot(-1,x_1-1,x_2-1)+1\cdot(0,x_1-1,x_2-1)+1\cdot(x_1-1,-1,x_2-1)-1\cdot(x_1-1,x_2-1,-1)$$
$$-1\cdot(-1,x_1-1,x_2)+1\cdot(0,x_1-1,x_2)+1\cdot(x_1-1,-1,x_2)-1\cdot(x_1-1,0,x_2)+1\cdot(x_1-1,x_2-1,1)$$

Finally we get the final 2-chain that corresponds to the boundary of the 3-chain $C_1(x) + C_2(x) + C_3(x) + C_4(x)$:

$$=1\cdot(1,x_1,x_2-1)+1\cdot(x_1,0,x_2-1)-1\cdot(x_1,1,x_2-1)-1\cdot(x_1,x_2,-1)+1\cdot(x_1,x_2,0)-1\cdot(-1,x_1,x_2-1)-1\cdot(x_1-1,1,x_2-1)$$
$$-1\cdot(x_1-1,x_2,-1)+1\cdot(x_1-1,x_2,0)-1\cdot(-1,x_1-1,x_2-1)+1\cdot(0,x_1-1,x_2-1)+1\cdot(x_1-1,-1,x_2-1)-1\cdot(x_1-1,x_2-1,-1)$$
$$-1\cdot(-1,x_1-1,x_2)+1\cdot(0,x_1-1,x_2)+1\cdot(x_1-1,-1,x_2)-1\cdot(x_1-1,0,x_2)+1\cdot(x_1-1,x_2-1,1)$$

Now, we will proceed to mention one standard property of ∂ operator.

**Theorem 4.1 [Spivak65]:** *If c is an n-chain in a closed set A, then ∂(∂(c)) = 0.*

We will sketch the proof given in [Spivak65]. Let $i \le j$ and consider $(I^n_{(i,\alpha)})_{(j,\beta)}$. If $x \in [0,1]^{n-2}$, then, we have

$$(I^n_{(i,\alpha)})_{(j,\beta)}(x) = I^n_{(i,\alpha)}(I^{n-1}_{(j,\beta)}(x)) = I^n_{(i,\alpha)}(x_1,...,x_{j-1},\beta,x_j,...,x_{n-2}) = I^n(x_1,...,x_{i-1},\alpha,x_i,...,x_{j-1},\beta,x_j,...,x_{n-2})$$

Similarly

$$(I^n_{(j+1,\beta)})_{(i,\alpha)}(x) = I^n_{(j+1,\beta)}(I^{n-1}_{(i,\alpha)}(x)) = I^n_{(j+1,\beta)}(x_1,...,x_{i-1},\alpha,x_i,...,x_{n-2}) = I^n(x_1,...,x_{i-1},\alpha,x_i,...,x_{j-1},\beta,x_j,...,x_{n-2})$$

Thus $(I^n_{(i,\alpha)})_{(j,\beta)}(x) = (I^n_{(j+1,\beta)})_{(i,\alpha)}(x)$ for $i \le j$. It follows for any singular general nD hyper-box c that $(c_{(i,\alpha)})_{(j,\beta)} = (c_{(j+1,\beta)})_{(i,\alpha)}$ when $i \le j$. Hence

$$\partial(\partial(c)) = \partial\left(\sum_{i=1}^{n}\left(\sum_{\alpha=0,1}(-1)^{i+\alpha}\cdot c_{(i,\alpha)}\right)\right) = \sum_{i=1}^{n}\left(\sum_{\alpha=0,1}\left(\sum_{j=1}^{n-1}\left(\sum_{\beta=0,1}(-1)^{i+\alpha+j+\beta}(c_{(i,\alpha)})_{(j,\beta)}\right)\right)\right)$$

In this sum $(c_{(i,\alpha)})_{(j,\beta)}$ and $(c_{(j+1,\beta)})_{(i,\alpha)}$ have opposite orientations. Therefore all terms cancel out in pairs and ∂(∂(c)) = 0. Since the Theorem is true for any singular general nD hyper-box, it is also true for any n-chain [Spivak65].

The last Theorem is actually the foundation of an important field in topology: Homology Theory ([Agoston05] & [Crossley05]).

## 4.3. Linking k-chains with Topological Local Analysis of Odd Edges

Now, we will apply the definitions presented in previous section in a new direction. We will take advantage of the fact that we can define equivalence relations in order to classify the (n-1)D cells in the boundary of a combination of hyper-boxes[2]. Our objective is to define in a precise manner and in unambiguously way which (n-1)D cells are we considering in our analysis. Some definitions previously presented in **Chapter 3** will be redefined according our new framework. Before going any further, we start by stating the following

**Definition 4.11 [Jonas95]:** *Consider a set* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ *of linearly independent vectors such that*

$$\boldsymbol{v}_i = (0, \ldots, 0, \gamma_i, 0, \ldots, 0),\ \gamma_i \in \mathbb{R}^+,\ i = 1, \ldots, n$$

*We define* <u>*a lattice*</u> $\underline{L^n_{(\gamma_1,...,\gamma_n)}}$ <u>*in*</u> $\underline{\mathbb{R}^n}$ *as the set of points defined by* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ *in the following way*:

$$L^n_{(\gamma_1,...,\gamma_n)} = \left\{ p \in \mathbb{R}^n : p = \sum_{i=1}^{n} p_i \boldsymbol{v}_i,\ \ p_i \in \mathbb{Z},\ \ i = 1,...,n \right\}$$

Consider the linearly independent vectors in $\mathbb{R}^2$ $\boldsymbol{v}_1 = (3, 0)$, $\boldsymbol{v}_2 = (0, 1)$. Such vectors define the lattice

$$L^2_{(3,1)} = \left\{ (x, y) \in \mathbb{R}^2 : (x, y) = p_1(3,0) + p_2(0,1),\ \ p_1, p_2 \in \mathbb{Z} \right\}$$

**Figure 4.5** shows some of the points in it.

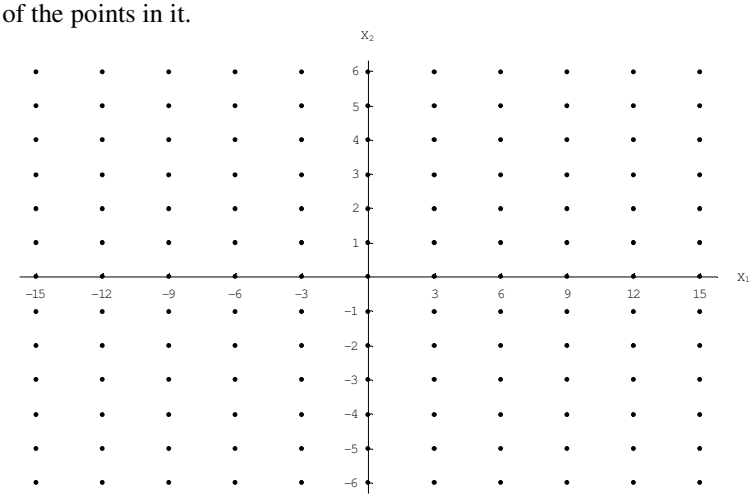



**Figure 4.5.** A view of the points in the lattice $L^2_{(3,1)}$

[2] The idea for partitioning the (n-1)D cells on the boundary of a combination of hyper-boxes was originated by personal communications with Guillermo Romero-Meléndez, PhD. (UDLA, Mathematics and Actuarial Sciences Department).

**Definition 4.12:** *Consider the lattice* $L^n_{(\gamma_1,\ldots,\gamma_n)}$. *Let a* <u>*main positive edge*</u> *in* $\mathbb{R}^n$ *be the function*

$$c^{(1,j,0)}: \ [0,1] \ \rightarrow \ \{\underbrace{(0,\ldots,0,x_j,0,\ldots,0)}_{n} \in \mathbb{R}^n : x_j \in [0,\gamma_j],\ 1 \le j \le n\}$$

$$x_1 \ \sim \ c^{(1,j,0)}(x_1) = \underbrace{(\underbrace{0,\ldots,0}_{j-1}, x_1 \cdot \gamma_j, 0,\ldots,0)}_{n}$$

*And let a* <u>*main negative edge*</u> *in* $\mathbb{R}^n$ *be the function*

$$c^{(1,j,1)}: \ [0,1] \ \rightarrow \ \{\underbrace{(0,\ldots,0,-x_j,0,\ldots,0)}_{n} \in \mathbb{R}^n : x_j \in [0,\gamma_j],\ 1 \le j \le n\}$$

$$x_1 \ \sim \ c^{(1,j,1)}(x_1) = \underbrace{(\underbrace{0,\ldots,0}_{j-1}, -x_1 \cdot \gamma_j, 0,\ldots,0)}_{n}$$

*for* $1 \le j \le n$.

For example, under the lattice $L^3_{(1,1,1)}$ we have the six main edges in 3D space:

- $c^{(1,1,0)}(x_1) = (x_1, 0, 0)$
- $c^{(1,1,1)}(x_1) = (-x_1, 0, 0)$
- $c^{(1,2,0)}(x_1) = (0, x_1, 0)$
- $c^{(1,2,1)}(x_1) = (0, -x_1, 0)$
- $c^{(1,3,0)}(x_1) = (0, 0, x_1)$
- $c^{(1,3,1)}(x_1) = (0, 0, -x_1)$

Consider a lattice $L^n_{(\gamma_1,\ldots,\gamma_n)}$. Starting from this point, when the term *general singular nD hyper-box* is referred, we will assume that we are considering one of the following $2^n$ functions:

$$c: \ [0,1]^n \ \rightarrow \ [\gamma_1 a_1,\ \gamma_1(a_1+1)] \times \ldots \times [\gamma_n a_n,\ \gamma_n(a_n+1)]$$

$$x \ \sim \ c(x) = (\gamma_1(x_1+a_1), \ldots, \gamma_n(x_n+a_n))$$

Where $a = (a_1, \ldots, a_n)$ such that $a_i \in \{-1,0\}$, $1 \le i \le n$. The nD hyper-boxes we are considering have one vertex in the origin because we will analyze the topology and geometry around that point from a local point of view (however, when we deal with global topology and geometry of the nD-OPP's we will consider more general nD hyper-boxes that not require to be attached to the origin).

For example, consider the set of linearly independent vectors in $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$ and $\boldsymbol{v}_3 = (0, 0, 1)$ that define the lattice $L^3_{(1,1,1)}$ in $\mathbb{R}^3$. Under these conditions we have the well known possible eight general singular 3D hyper-boxes which are shown in **Table 4.1**.

In the propositions presented in the remaining of this section, we are assuming that our objects are defined under the same lattice $L^n_{(\gamma_1,\ldots,\gamma_n)}$.

**Definition 4.13:** *Let* $c^{(1,j,\beta)}$ *be a main edge and let c be a general singular kD hyper-box,* $1 \le k \le n$. *We will say that* $c^{(1,j,\beta)}$ *is* <u>*adjacent*</u> *c if and only if*

$$c^{(1,j,\beta)}([0,1]) \subseteq c([0,1]^k)$$

**Definition 4.14:** *A collection* $c_1, c_2, \ldots, c_k$, $1 \le k \le 2^n$, *of general singular nD hyper-boxes is a* <u>*combination of nD hyper-boxes*</u> *if and only if*

$$\left[\bigcap_{\alpha=1}^{k} c_\alpha([0,1]^n) = \underbrace{(0,\ldots,0)}_{n}\right] \wedge \left[\left(\forall i, j,\ i \ne j,\ 1 \le i, j \le k\right)\left(c_i([0,1]^n) \ne c_j([0,1]^n)\right)\right]$$

In the above definition the left side of the conjunction establishes that the intersection between all the nD general singular hyper-boxes is the origin, while the right side establishes that there are not overlapping nD hyper-boxes.

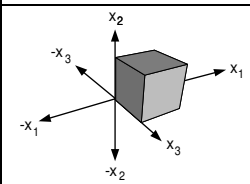
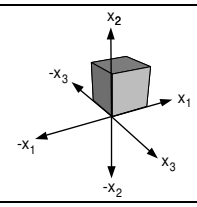
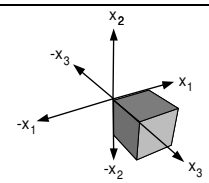
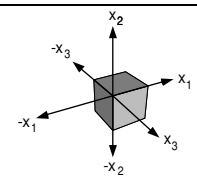
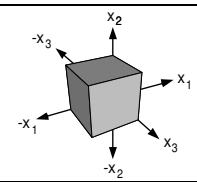
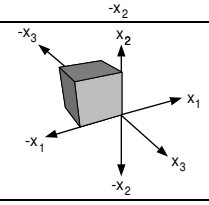
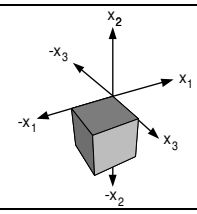
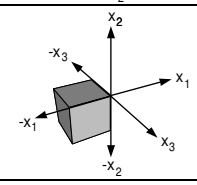
| Vector *a* | General Singular Cube | |
|---|---|---|
| (0,0,0) | $c_1: \ [0,1]^3 \ \rightarrow \ [0,1]\times[0,1]\times[0,1]$<br>$x \ \sim \ c_1(x) = (1\cdot(x_1+0), 1\cdot(x_2+0), 1\cdot(x_3+0)) = (x_1, x_2, x_3)$ |  |
| (0,0,-1) | $c_2: \ [0,1]^3 \ \rightarrow \ [0,1]\times[0,1]\times[-1,0]$<br>$x \ \sim \ c_2(x) = (1\cdot(x_1+0), 1\cdot(x_2+0), 1\cdot(x_3-1)) = (x_1, x_2, x_3-1)$ |  |
| (0,-1,0) | $c_3: \ [0,1]^3 \ \rightarrow \ [0,1]\times[-1,0]\times[0,1]$<br>$x \ \sim \ c_3(x) = (1\cdot(x_1+0), 1\cdot(x_2-1), 1\cdot(x_3+0)) = (x_1, x_2-1, x_3)$ |  |
| (0,-1,-1) | $c_4: \ [0,1]^3 \ \rightarrow \ [0,1]\times[-1,0]\times[-1,0]$<br>$x \ \sim \ c_4(x) = (1\cdot(x_1+0), 1\cdot(x_2-1), 1\cdot(x_3-1)) = (x_1, x_2-1, x_3-1)$ |  |
| (-1,0,0) | $c_5: \ [0,1]^3 \ \rightarrow \ [-1,0]\times[0,1]\times[0,1]$<br>$x \ \sim \ c_5(x) = (1\cdot(x_1-1), 1\cdot(x_2+0), 1\cdot(x_3+0)) = (x_1-1, x_2, x_3)$ |  |
| (-1,0,-1) | $c_6: \ [0,1]^3 \ \rightarrow \ [-1,0]\times[0,1]\times[-1,0]$<br>$x \ \sim \ c_6(x) = (1\cdot(x_1-1), 1\cdot(x_2+0), 1\cdot(x_3-1)) = (x_1-1, x_2, x_3-1)$ |  |
| (-1,-1,0) | $c_7: \ [0,1]^3 \ \rightarrow \ [-1,0]\times[-1,0]\times[0,1]$<br>$x \ \sim \ c_7(x) = (1\cdot(x_1-1), 1\cdot(x_2-1), 1\cdot(x_3+0)) = (x_1-1, x_2-1, x_3)$ |  |
| (-1,-1,-1) | $c_8: \ [0,1]^3 \ \rightarrow \ [-1,0]\times[-1,0]\times[-1,0]$<br>$x \ \sim \ c_8(x) = (1\cdot(x_1-1), 1\cdot(x_2-1), 1\cdot(x_3-1)) = (x_1-1, x_2-1, x_3-1)$ |  |

**Table 4.1.** The eight possible general singular 3D hyper-boxes with a vertex in the origin of the lattice $L^3_{(1,1,1)}$.

**Definition 4.15:** *Let $c^{(1,j,\beta)}$ be a main edge. Let $c_1, c_2, \ldots, c_k$, $1 \le k \le 2^n$, be a combination of general singular nD hyper-boxes. We will say that $c^{(1,j,\beta)}$ is an* <u>*Odd Adjacency Edge*</u>*, or just an* <u>*Odd Edge*</u>*, if and only if $c^{(1,j,\beta)}$ is adjacent to an odd number of general singular nD hyper-boxes of c. Conversely, if $c^{(1,j,\beta)}$ is adjacent to an even number of general singular nD hyper-boxes, then it will be called an* <u>*Even Adjacency Edge*</u>*, or just* <u>*Even Edge*</u>*.*

**Definition 4.16:** *Let* $c^{(1,j,\beta)}$ *be a main edge.* $\overline{c^{(1,j,\beta')}}$ *will denote to the* <u>*collinear main edge opposed to*</u> $c^{(1,j,\beta)}$ *such that if* $\beta = 0$ *then* $\beta' = 1$*, or if* $\beta = 1$ *then* $\beta' = 0$*.*

For example, under lattice $L^3_{(1,1,1)}$, for each main edge in $\mathbb{R}^3$ we will have:

- $c^{(1,1,0)}(x_1) = (x_1, 0, 0) \Rightarrow \overline{c^{(1,1,0)}}(x_1) = c^{(1,1,1)}(x_1) = (-x_1, 0, 0)$
- $c^{(1,1,1)}(x_1) = (-x_1, 0, 0) \Rightarrow \overline{c^{(1,1,1)}}(x_1) = c^{(1,1,0)}(x_1) = (x_1, 0, 0)$
- $c^{(1,2,0)}(x_1) = (0, x_1, 0) \Rightarrow \overline{c^{(1,2,0)}}(x_1) = c^{(1,2,1)}(x_1) = (0, -x_1, 0)$
- $c^{(1,2,1)}(x_1) = (0, -x_1, 0) \Rightarrow \overline{c^{(1,2,1)}}(x_1) = c^{(1,2,0)}(x_1) = (0, x_1, 0)$
- $c^{(1,3,0)}(x_1) = (0, 0, x_1) \Rightarrow \overline{c^{(1,3,0)}}(x_1) = c^{(1,3,1)}(x_1) = (0, 0, -x_1)$
- $c^{(1,3,1)}(x_1) = (0, 0, -x_1) \Rightarrow \overline{c^{(1,3,1)}}(x_1) = c^{(1,3,0)}(x_1) = (0, 0, x_1)$

We reconsider some notation presented previously in **Chapter 3**. We defined to $\mathbb{R}_i^+$ as the subspace defined by the positive part of $x_i$-axis and the remaining axes of the nD space; while $\mathbb{R}_i^-$ denotes to the subspace defined by the negative part of $x_i$-axis and the remaining axes of the nD space. $x_i^+$ denotes to the positive part of $x_i$-axis while $x_i^-$ denotes to the negative part of $x_i$-axis. Finally, in a combination of nD hyper-boxes c, Γ(c) denotes the number of hyper-boxes in c.

**Lemma 4.1:** *Let c be a combination of general singular nD hyper-boxes. If combination c has an axis* $x_i$*,* $1 \le i \le n$*, where there is exactly an odd edge, then c has an odd number of nD hyper-boxes, i.e.,* Γ(c) *is an odd number.*
Proof:
Let $c^{(1,i,\beta)}$ be the referred odd edge. Hence, by definition, the number $n_1$ of nD hyper-boxes in $\mathbb{R}_i^+$ (or $\mathbb{R}_i^-$, according to the value of $\beta$) is odd. Because by hypothesis $\overline{c^{(1,i,\beta)}}$ is an even edge, then the number $n_2$ of its incident nD hyper-boxes is even. Therefore

$$n_1 + n_2 = \Gamma(c) \text{ is an odd number.}$$

◪

**Theorem 4.2:** *Let c be a combination of general singular nD hyper-boxes. In c there are exactly n linearly independent odd edges, which are incident to the origin of the coordinate system in the combination, if and only if combination* ***c*** *has an odd number of nD hyper-boxes.*
Proof:
⇒)
Consider $x_i$-axis, $1 \le i \le n$, in which one of the odd edges, namely $c^{(1,i,\beta)}$, is embedded. By hypothesis $c^{(1,i,\beta)}$ has an odd number $n_1^i$ of nD hyper-boxes which are incident to it. Such nD hyper-boxes will be embedded in $\mathbb{R}_i^+$ or $\mathbb{R}_i^-$ according to the referred odd edge. Let $n_2^i$ be the even number of nD hyper-boxes that are incident to the even edge $\overline{c^{(1,i,\beta)}}$. Hence, by **Lemma 4.1**, $n_1^i + n_2^i = \Gamma(c)$ is an odd number. By applying the same procedure to the remaining n-1 axes we obtain the same result.
$\therefore\ n_1^i + n_2^i = \Gamma(c)$ is an odd number $\forall i$, $1 \le i \le n$.
⇐)
Consider $x_i$-axis, $1 \le i \le n$. Because Γ(c) is an odd number, the hyper-boxes in the combination c are distributed in such way that in $\mathbb{R}_i^+$ (or $\mathbb{R}_i^-$) there are an odd number of hyper-boxes while in $\mathbb{R}_i^-$ (or $\mathbb{R}_i^+$) there are an even number of nD hyper-boxes. Therefore, by definition, there is an odd edge incident to the origin in $\mathbb{R}_i^+$ (or $\mathbb{R}_i^-$). By applying this reasoning to the remaining n-1 axes we identify in each one an odd edge incident to the origin. By this way we have identified n linearly independent odd edges incident to the origin in the combination c.

◪

**Corollary 4.1:** *Let c be a combination of general singular nD hyper-boxes. In c there are n pairs of collinear odd edges or collinear even edges, incident to the origin, if and only if combination c has an even number of hyper-boxes.*
Proof:
This proposition is the counterreciprocal of **Theorem 4.2** ($p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$). ◪

**Definition 4.17:** *Let* $c^{(1,j,\beta)}$ *be a main edge and let* $(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}$ *be an oriented cell of the general singular nD hyper-box c. We will say that* $c^{(1,j,\beta)}$ *is perpendicular to* $c\circ I^n_{(i,\alpha)}$ *if and only if* $c^{(1,j,\beta)}$ *is adjacent to c and if the intersection between* $c^{(1,j,\beta)}$ *and* $c\circ I^n_{(i,\alpha)}$ *is the origin, i.e.,*

$$(c^{(1,j,\beta)}([0,1])\subseteq c([0,1]^n))\wedge(c^{(1,j,\beta)}([0,1])\cap c(I^n_{(i,\alpha)}([0,1]^n))=(\underbrace{0,\ldots,0}_{n}))$$

**Definition 4.18:** *Let* $c^{(1,j,\beta)}$ *be a main edge. Let* $c_1, c_2, \ldots, c_k$, $1\le k\le 2^n$, *be a combination of general singular nD hyper-boxes with oriented cells* $(-1)^{i+\alpha}\cdot c_1\circ I^n_{(i,\alpha)}, \ldots, (-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}$ *respectively,* $1\le i\le n$, $\alpha\in\{0,1\}$. *We define the relation* $R_P^{c^{(1,j,\beta)}}$ *as:*

$$c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k'}\circ I^n_{(i',\alpha')}\Leftrightarrow([\,c^{(1,j,\beta)}\text{ is perpendicular to }c_k\circ I^n_{(i,\alpha)}]\wedge[\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k'}\circ I^n_{(i',\alpha')}])$$

*where* $1\le k'\le k$, $1\le i'\le n$, $\alpha'\in\{0,1\}$.

**Theorem 4.3:** *Relation* $R_P^{c^{(1,j,\beta)}}$ *is an equivalence relation.*

Proof:

Let $(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}$, $(-1)^{i+\alpha}\cdot c_{k'}\circ I^n_{(i',\alpha')}$ and $(-1)^{i+\alpha}\cdot c_{k''}\circ I^n_{(i'',\alpha'')}$ be oriented cells of general singular nD hyper-boxes, $1\le i,i',i''\le n$, $\alpha,\alpha',\alpha''\in\{0,1\}$.

The following properties are satisfied:

- Reflexivity: $(\forall(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}, 1\le i\le n,\alpha\in\{0,1\})(c\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c\circ I^n_{(i,\alpha)})$
- Symmetry:

  If $c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k'}\circ I^n_{(i',\alpha')}$

  $\Rightarrow([\,c^{(1,j,\beta)}\text{ is perpendicular to }c_k\circ I^n_{(i,\alpha)}]\wedge[\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k'}\circ I^n_{(i',\alpha')}])$

  $\Rightarrow([\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k'}\circ I^n_{(i',\alpha')}]\wedge[\,c^{(1,j,\beta)}\text{ is perpendicular to }c_k\circ I^n_{(i,\alpha)}])$

  $\Rightarrow c_{k'}\circ I^n_{(i',\alpha')}\ R_P^{c^{(1,j,\beta)}}\ c_k\circ I^n_{(i,\alpha)}$

  $\therefore(\forall(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)},\ 1\le i\le n,\ \alpha\in\{0,1\})(\forall(-1)^{i+\alpha}\cdot c_{k'}\circ I^n_{(i',\alpha')},\ 1\le i'\le n,\ \alpha'\in\{0,1\})$

  $(c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k'}\circ I^n_{(i',\alpha')}\Rightarrow c_{k'}\circ I^n_{(i',\alpha')}\ R_P^{c^{(1,j,\beta)}}\ c_k\circ I^n_{(i,\alpha)})$
- Transitivity:

  If $c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k'}\circ I^n_{(i',\alpha')}\wedge c_{k'}\circ I^n_{(i',\alpha')}\ R_P^{c^{(1,j,\beta)}}\ c_{k''}\circ I^n_{(i'',\alpha'')}$

  $\Rightarrow([\,c^{(1,j,\beta)}\text{ is perpendicular to }c_k\circ I^n_{(i,\alpha)}]\wedge[\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k'}\circ I^n_{(i',\alpha')}])\wedge$

  $([\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k'}\circ I^n_{(i',\alpha')}]\wedge[\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k''}\circ I^n_{(i'',\alpha'')}])$

  $\Rightarrow([\,c^{(1,j,\beta)}\text{ is perpendicular to }c_k\circ I^n_{(i,\alpha)}]\wedge[\,c^{(1,j,\beta)}\text{ is perpendicular to }c_{k''}\circ I^n_{(i'',\alpha'')}])$

  $\Rightarrow c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k''}\circ I^n_{(i'',\alpha'')}$

  $\therefore(\forall(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)},\ 1\le i\le n,\ \alpha\in\{0,1\})(\forall(-1)^{i+\alpha}\cdot c_{k'}\circ I^n_{(i',\alpha')},\ 1\le i'\le n,\ \alpha'\in\{0,1\})$

  $(\forall(-1)^{i+\alpha}\cdot c_{k''}\circ I^n_{(i'',\alpha'')},\ 1\le i''\le n,\ \alpha''\in\{0,1\})$

  $(c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k'}\circ I^n_{(i',\alpha')}\wedge c_{k'}\circ I^n_{(i',\alpha')}\ R_P^{c^{(1,j,\beta)}}\ c_{k''}\circ I^n_{(i'',\alpha'')}\Rightarrow c_k\circ I^n_{(i,\alpha)}\ R_P^{c^{(1,j,\beta)}}\ c_{k''}\circ I^n_{(i'',\alpha'')})$

$\therefore$ Relation $R_P^{c^{(1,j,\beta)}}$ is an equivalence relation. ◪

**Definition 4.19:** *Consider equivalence relation* $R_P^{c^{(1,j,\beta)}}$. *The set*

$$[(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}=\{(-1)^{i'+\alpha'}\cdot c_{k'}\circ I^n_{(i',\alpha')}: c_{k'}\circ I^n_{(i',\alpha')}\ R_P^{c^{(1,j,\beta)}}\ c_k\circ I^n_{(i,\alpha)}\}$$

*is the equivalence class under* $R_P^{c^{(1,j,\beta)}}$ *of the oriented cell* $(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}$ *induced by the main edge* $c^{(1,j,\beta)}$ *and whose representative is* $(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}$.

For example, consider the following combination of 3D boxes under lattice $L^3_{(1,1,1)}$ (see **Figure 4.6**):

- $c_1: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3)\in\mathbb{R}^3 : 0\le x_1\le 1, 0\le x_2\le 1, 0\le x_3\le 1\}$
  $x \ \sim \ c_1(x)=(x_1,x_2,x_3)$
- $c_2: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3)\in\mathbb{R}^3 : 0\le x_1\le 1, -1\le x_2\le 0, 0\le x_3\le 1\}$
  $x \ \sim \ c_2(x)=(x_1,x_2-1,x_3)$
- $c_3: \ [0,1]^3 \ \rightarrow \ \{(x_1,x_2,x_3)\in\mathbb{R}^3 : -1\le x_1\le 0, -1\le x_2\le 0, -1\le x_3\le 0\}$
  $x \ \sim \ c_3(x)=(x_1-1,x_2-1,x_3-1)$

Whose corresponding oriented cells are shown in **Table 4.2**.

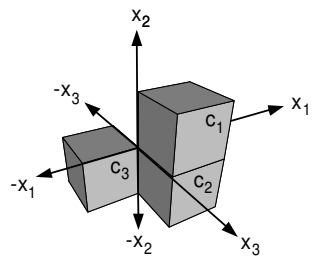



**Figure 4.6.** A 3D combination of general singular cubes (see text for details).

| $c_1$'s oriented cells | $c_2$'s oriented cells | $c_3$'s oriented cells |
|---|---|---|
| $(-1)^{1+0}\cdot c_1(I^3_{(1,0)}(x))=-1\cdot(0,x_1,x_2)$ | $(-1)^{1+0}\cdot c_2(I^3_{(1,0)}(x))=-1\cdot(0,x_1-1,x_2)$ | $(-1)^{1+0}\cdot c_3(I^3_{(1,0)}(x))=-1\cdot(-1,x_1-1,x_2-1)$ |
| $(-1)^{1+1}\cdot c_1(I^3_{(1,1)}(x))=1\cdot(1,x_1,x_2)$ | $(-1)^{1+1}\cdot c_2(I^3_{(1,1)}(x))=1\cdot(1,x_1-1,x_2)$ | $(-1)^{1+1}\cdot c_3(I^3_{(1,1)}(x))=1\cdot(0,x_1-1,x_2-1)$ |
| $(-1)^{2+0}\cdot c_1(I^3_{(2,0)}(x))=1\cdot(x_1,0,x_2)$ | $(-1)^{2+0}\cdot c_2(I^3_{(2,0)}(x))=1\cdot(x_1,-1,x_2)$ | $(-1)^{2+0}\cdot c_3(I^3_{(2,0)}(x))=1\cdot(x_1-1,-1,x_2-1)$ |
| $(-1)^{2+1}\cdot c_1(I^3_{(2,1)}(x))=-1\cdot(x_1,1,x_2)$ | $(-1)^{2+1}\cdot c_2(I^3_{(2,1)}(x))=-1\cdot(x_1,0,x_2)$ | $(-1)^{2+1}\cdot c_3(I^3_{(2,1)}(x))=-1\cdot(x_1-1,0,x_2-1)$ |
| $(-1)^{3+0}\cdot c_1(I^3_{(3,0)}(x))=-1\cdot(x_1,x_2,0)$ | $(-1)^{3+0}\cdot c_2(I^3_{(3,0)}(x))=-1\cdot(x_1,x_2-1,0)$ | $(-1)^{3+0}\cdot c_2(I^3_{(3,0)}(x))=-1\cdot(x_1-1,x_2-1,-1)$ |
| $(-1)^{3+1}\cdot c_1(I^3_{(3,1)}(x))=1\cdot(x_1,x_2,1)$ | $(-1)^{3+1}\cdot c_2(I^3_{(3,1)}(x))=1\cdot(x_1,x_2-1,1)$ | $(-1)^{3+1}\cdot c_3(I^3_{(3,1)}(x))=1\cdot(x_1-1,x_2-1,0)$ |

**Table 4.2.** Oriented cells of a 3D combination of hyper-boxes (see text for details).

Now, we identify the cells that are perpendicular to each main edge in $\mathbb{R}^3$. Based in this information we build the equivalence classes induced by these edges and cells. See **Table 4.3**.

**Definition 4.20:** *Consider equivalence class* $[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$. *We define the set* $\underline{\eta^{c^{(1,j,\beta)}}}$ *as follows:*

$$\eta^{c^{(1,j,\beta)}}=\left\{\begin{array}{l}(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}\in[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}: \\ (-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}+(-1)^{i'+\alpha'}\cdot c\circ I^n_{(i',\alpha')}=0,\ (-1)^{i'+\alpha'}\cdot c\circ I^n_{(i',\alpha')}\in[(-1)^{i'+\alpha'}\cdot c\circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\end{array}\right\}$$

The set $\eta^{c^{(1,j,\beta)}}$ contains the cells of $[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$ that are included also in $[(-1)^{i'+\alpha'}\cdot c\circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}$ (the set of cells that are perpendicular to the collinear opposite edge of $c^{(1,j,\beta)}$) but with opposite orientation. Such cells are not included in ∂(c).

**Definition 4.21:** *Consider equivalence class* $[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$. *We define the set* $\underline{\wp^{c^{(1,j,\beta)}}}$ *as follows:*

$$\wp^{c^{(1,j,\beta)}}=[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\setminus\eta^{c^{(1,j,\beta)}}$$

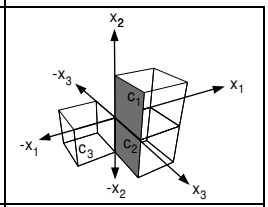
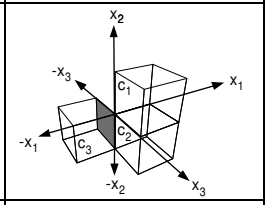
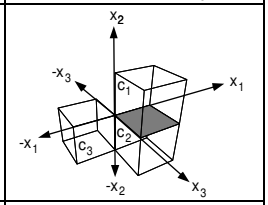
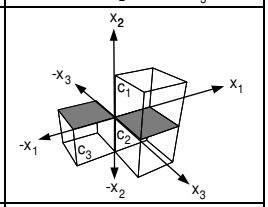
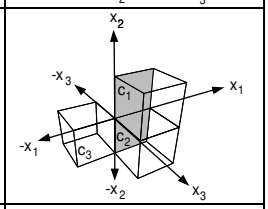
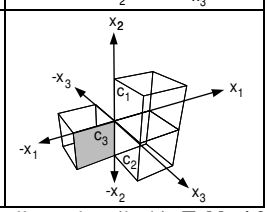
| Main Edge | Perpendicular 2DCells | Equivalence Class | |
|---|---|---|---|
| $c^{(1,1,0)}(x_1)=(x_1,0,0)$ | $c_1(I^3_{(1,0)}(x))=(0,x_1,x_2)$<br>$c_2(I^3_{(1,0)}(x))=(0,x_1-1,x_2)$ | $\left[-1\cdot c_1(I^3_{(1,0)}(x))\right]^{R_P}_{c^{(1,1,0)}}$<br>$=\{-1\cdot(0,x_1,x_2),-1\cdot(0,x_1-1,x_2)\}$ |  |
| $c^{(1,1,1)}(x_1)=(-x_1,0,0)$ | $c_3(I^3_{(1,1)}(x))=(0,x_1-1,x_2-1)$ | $\left[1\cdot c_3(I^3_{(1,1)}(x))\right]^{R_P}_{c^{(1,1,1)}}$<br>$=\{1\cdot(0,x_1-1,x_2-1)\}$ |  |
| $c^{(1,2,0)}(x_1)=(0,x_1,0)$ | $c_1(I^3_{(2,0)}(x))=(x_1,0,x_2)$ | $\left[1\cdot c_1(I^3_{(2,0)}(x))\right]^{R_P}_{c^{(1,2,0)}}$<br>$=\{1\cdot(x_1,0,x_2)\}$ |  |
| $c^{(1,2,1)}(x_1)=(0,-x_1,0)$ | $c_2(I^3_{(2,1)}(x))=(x_1,0,x_2)$<br>$c_3(I^3_{(2,1)}(x))=(x_1-1,0,x_2-1)$ | $\left[-1\cdot c_2(I^3_{(2,1)}(x))\right]^{R_P}_{c^{(1,2,1)}}$<br>$=\{-1\cdot(x_1,0,x_2),-1\cdot(x_1-1,0,x_2-1)\}$ |  |
| $c^{(1,3,0)}(x_1)=(0,0,x_1)$ | $c_1(I^3_{(3,0)}(x))=(x_1,x_2,0)$<br>$c_2(I^3_{(3,0)}(x))=(x_1,x_2-1,0)$ | $\left[-1\cdot c_2(I^3_{(3,0)}(x))\right]^{R_P}_{c^{(1,3,0)}}$<br>$=\{-1\cdot(x_1,x_2,0),-1\cdot(x_1,x_2-1,0)\}$ |  |
| $c^{(1,3,1)}(x_1)=(0,0,-x_1)$ | $c_3(I^3_{(3,1)}(x))=(x_1-1,x_2-1,0)$ | $\left[1\cdot c_3(I^3_{(3,1)}(x))\right]^{R_P}_{c^{(1,3,1)}}$<br>$=\{1\cdot(x_1-1,x_2-1,0)\}$ |  |

**Table 4.3.** Computing the Equivalence Classes under relation $R_P$ of the 3D combination whose cells are described in **Table 4.2**.

The set $\wp^{c^{(1,j,\beta)}}$ contains all the cells in $[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$ except those such that are included, with opposite orientation, in $[(-1)^{i'+\alpha'}\cdot c\circ I^n_{(i',\alpha')}]^{R_P}_{c^{(1,j,\beta)}}$. Consider for example the 3D combination whose cells are described in **Table 4.2.** For each one of its equivalence classes we have its corresponding sets $\wp^{c^{(1,j,\beta)}}$ which are described in **Table 4.4**.

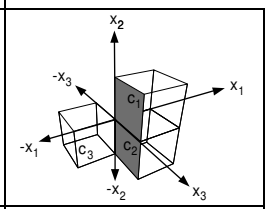
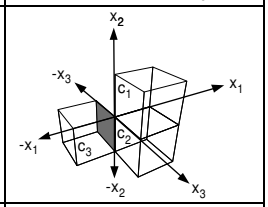
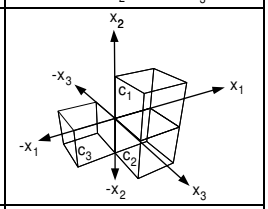
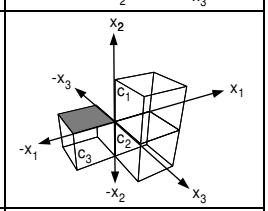
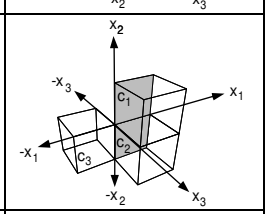
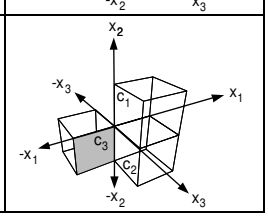
| $[(-1)^{i+\alpha}\cdot c\circ I^{n}_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$ | $\eta^{c^{(1,j,\beta)}}$ | $\wp^{c^{(1,j,\beta)}}=[(-1)^{i+\alpha}\cdot c\circ I^{n}_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\setminus\eta^{c^{(1,j,\beta)}}$ | |
|---|---|---|---|
| $\left[-1\cdot c_1(I^3_{(1,0)}(x))\right]^{R_P}_{c^{(1,1,0)}}$ $=\{-1\cdot(0,x_1,x_2),-1\cdot(0,x_1-1,x_2)\}$ | $\eta^{c^{(1,1,0)}}=\varnothing$ | $\wp^{c^{(1,1,0)}}=\left[-1\cdot c_1(I^3_{(1,0)}(x))\right]^{R_P}_{c^{(1,1,0)}}\setminus\varnothing$ $=\{-1\cdot(0,x_1,x_2),-1\cdot(0,x_1-1,x_2)\}$ |   |
| $\left[1\cdot c_3(I^3_{(1,1)}(x))\right]^{R_P}_{c^{(1,1,1)}}$ $=\{1\cdot(0,x_1-1,x_2-1)\}$ | $\eta^{c^{(1,1,1)}}=\varnothing$ | $\wp^{c^{(1,1,1)}}=\left[1\cdot c_3(I^3_{(1,1)}(x))\right]^{R_P}_{c^{(1,1,1)}}\setminus\varnothing$ $=\{1\cdot(0,x_1-1,x_2-1)\}$ |   |
| $\left[1\cdot c_1(I^3_{(2,0)}(x))\right]^{R_P}_{c^{(1,2,0)}}$ $=\{1\cdot(x_1,0,x_2)\}$ | $\eta^{c^{(1,2,0)}}=\{1\cdot(x_1,0,x_2)\}$ | $\wp^{c^{(1,2,0)}}=\left[1\cdot c_1(I^3_{(2,0)}(x))\right]^{R_P}_{c^{(1,2,0)}}\setminus\{1\cdot(x_1,0,x_2)\}$ $=\varnothing$ |   |
| $\left[-1\cdot c_2(I^3_{(2,1)}(x))\right]^{R_P}_{c^{(1,2,1)}}$ $=\{-1\cdot(x_1,0,x_2),-1\cdot(x_1-1,0,x_2-1)\}$ | $\eta^{c^{(1,2,1)}}=\{-1\cdot(x_1,0,x_2)\}$ | $\wp^{c^{(1,2,1)}}=\left[-1\cdot c_2(I^3_{(2,1)}(x))\right]^{R_P}_{c^{(1,2,1)}}\setminus\{-1\cdot(x_1,0,x_2)\}$ $=\{-1\cdot(x_1-1,0,x_2-1)\}$ |   |
| $\left[-1\cdot c_2(I^3_{(3,0)}(x))\right]^{R_P}_{c^{(1,3,0)}}$ $=\{-1\cdot(x_1,x_2,0),-1\cdot(x_1,x_2-1,0)\}$ | $\eta^{c^{(1,3,0)}}=\varnothing$ | $\wp^{c^{(1,3,0)}}=\left[-1\cdot c_2(I^3_{(3,0)}(x))\right]^{R_P}_{c^{(1,3,0)}}\setminus\varnothing$ $=\{-1\cdot(x_1,x_2,0),-1\cdot(x_1,x_2-1,0)\}$ |   |
| $\left[1\cdot c_3(I^3_{(3,1)}(x))\right]^{R_P}_{c^{(1,3,1)}}$ $=\{1\cdot(x_1-1,x_2-1,0)\}$ | $\eta^{c^{(1,3,1)}}=\varnothing$ | $\wp^{c^{(1,3,1)}}=\left[-1\cdot c_1(I^3_{(3,0)}(x))\right]^{R_P}_{c^{(1,3,1)}}\setminus\varnothing$ $=\{1\cdot(x_1-1,x_2-1,0)\}$ |   |

**Table 4.4.** Computing the sets $\wp^{c^{(1,j,\beta)}}$ for the 3D combination of cells described in **Table 4.2**.

Although the following three propositions we will list are obvious, we present them as Properties because they will be very useful in proving some next results. It is easy to see that the sets $\wp^{c^{(1,j,\beta)}}$ and $\wp^{\overline{c^{(1,j,\beta)}}}$ do not have common cells (even ignoring orientations), hence

**Property 4.1:** *Sets* $\wp^{c^{(1,j,\beta)}}$ *and* $\wp^{\overline{c^{(1,j,\beta)}}}$ *are disjoint sets, i.e.,* $\wp^{c^{(1,j,\beta)}}\cap\wp^{\overline{c^{(1,j,\beta)}}}=\varnothing$.

By ignoring orientations we have that the sets $\eta^{c^{(1,j,\beta)}}$ and $\eta^{\overline{c^{(1,j,\beta)}}}$ have exactly the same (n-1)D cells. Therefore

**Property 4.2:** $Card\left(\eta^{c^{(1,j,\beta)}}\right)=Card\left(\eta^{\overline{c^{(1,j,\beta)}}}\right)$

Finally, from the definition of $\wp^{c^{(1,j,\beta)}}$ we have that $\eta^{c^{(1,j,\beta)}} \subseteq [(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$. Hence

**Property 4.3:** $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}} \cap \eta^{c^{(1,j,\beta)}}\right) = Card\left(\eta^{c^{(1,j,\beta)}}\right)$

**Lemma 4.2:** *Consider a combination of nD hyper-boxes c. Let* $c^{(1,j,\beta)}$ *be a main edge on axis* $x_j$, $1 \le j \le n$, *and let* $\overline{c^{(1,j,\beta)}}$ *be its corresponding opposite collinear edge. Then, the total number of (n-1)D cells in* $\partial(c)$ *which are perpendicular to* $c^{(1,j,\beta)}$ *is given by*

$$Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - 2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$$

Proof:

Consider the following sets associated to $c^{(1,j,\beta)}$:

- $[(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$
- $\eta^{c^{(1,j,\beta)}}$
- $\wp^{c^{(1,j,\beta)}}$

The sets associated to $\overline{c^{(1,j,\beta)}}$ are:

- $[(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{\overline{c^{(1,j,\beta)}}}$
- $\eta^{\overline{c^{(1,j,\beta)}}}$
- $\wp^{\overline{c^{(1,j,\beta)}}}$

The set of all the cells included in ∂(c) which are perpendicular to $c^{(1,j,\beta)}$ is given by $\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}$.
We will show that

$$Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right) = Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - 2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$$

$$Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right)$$
$$= Card\left(\wp^{c^{(1,j,\beta)}}\right) + Card\left(\wp^{\overline{c^{(1,j,\beta)}}}\right) - Card\left(\wp^{c^{(1,j,\beta)}} \cap \wp^{\overline{c^{(1,j,\beta)}}}\right)$$

Because $\wp^{c^{(1,j,\beta)}}$ and $\wp^{\overline{c^{(1,j,\beta)}}}$ are disjoint sets (**Property 4.1**):

$$= Card\left(\wp^{c^{(1,j,\beta)}}\right) + Card\left(\wp^{\overline{c^{(1,j,\beta)}}}\right) - 0$$

By definition of $\wp^{c^{(1,j,\beta)}}$ and $\wp^{\overline{c^{(1,j,\beta)}}}$ we have that:

$$= Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}} \setminus \eta^{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}} \setminus \eta^{\overline{c^{(1,j,\beta)}}}\right)$$

By a well known result in Set Theory (Card(A \ B) = Card(A) – Card(A ∩ B)):

$$= \left[Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) - Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}} \cap \eta^{c^{(1,j,\beta)}}\right)\right] +$$
$$\left[Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}} \cap \eta^{\overline{c^{(1,j,\beta)}}}\right)\right]$$

By **Property 4.3**:

$$= \left[Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) - Card\left(\eta^{c^{(1,j,\beta)}}\right)\right] + \left[Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - Card\left(\eta^{\overline{c^{(1,j,\beta)}}}\right)\right]$$

Because **Property 4.2** states that $Card\left(\eta^{c^{(1,j,\beta)}}\right) = Card\left(\eta^{\overline{c^{(1,j,\beta)}}}\right)$ then:

$$= Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - 2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$$

Finally we have:

$$Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right) = Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - 2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$$

◪

**Lemma 4.3:** *Consider a combination of nD hyper-boxes c where there is an odd (even) edge $e_0$ on axis $x_j$, $1 \le j \le n$, such that its corresponding collinear edge is also odd (even). Then, the total number of (n-1)D cells in $\partial(c)$ which are perpendicular to $e_0$ is even.*

Proof:

The odd (even) edge $e_0$ will be denoted by main edge $c^{(1,j,\beta)}$ while $\overline{c^{(1,j,\beta)}}$ will denote to its collinear odd (even) edge. Consider the following sets associated to $c^{(1,j,\beta)}$:

- $[(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$
- $\eta^{c^{(1,j,\beta)}}$
- $\wp^{c^{(1,j,\beta)}}$

The sets associated to $\overline{c^{(1,j,\beta)}}$ are:

- $[(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{\overline{c^{(1,j,\beta)}}}$
- $\eta^{\overline{c^{(1,j,\beta)}}}$
- $\wp^{\overline{c^{(1,j,\beta)}}}$

We will show that $Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right)$ is even.

By **Lemma 4.2** we have that

$$Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right) = Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - 2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$$

If $c^{(1,j,\beta)}$ and $\overline{c^{(1,j,\beta)}}$ are both odd edges then:

- $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right)$ is an odd number.
- $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right)$ is an odd number.
- $2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$ is an even number.

If $c^{(1,j,\beta)}$ and $\overline{c^{(1,j,\beta)}}$ are both even edges then:

- $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right)$ is an odd number.
- $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right)$ is an odd number.
- $2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$ is an even number.

In both cases we conclude that $Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right)$ is an even number. ◪

**Theorem 4.4:** *Let c be a combination of nD hyper-boxes. In c exists exactly one odd edge $e_0$ on axis $x_j$, $1 \le j \le n$, if and only if the total number of (n-1)D cells in $\partial(c)$ which are perpendicular to $e_0$ is odd.*

Proof:

$\Rightarrow$)

Let the odd edge $e_0$ be denoted by main edge $c^{(1,j,\beta)}$, while $\overline{c^{(1,j,\beta)}}$ will denote to its corresponding collinear even edge. Consider the following sets associated to $c^{(1,j,\beta)}$:

- $[(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}$
- $\eta^{c^{(1,j,\beta)}}$
- $\wp^{c^{(1,j,\beta)}}$

The sets associated to $\overline{c^{(1,j,\beta)}}$ are:

- $[(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{\overline{c^{(1,j,\beta)}}}$
- $\eta^{\overline{c^{(1,j,\beta)}}}$
- $\wp^{\overline{c^{(1,j,\beta)}}}$

We will show that $Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right)$ is an odd number.

By **Lemma 4.2** we have that

$$Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right) = Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right) + Card\left([(-1)^{i'+\alpha'} \cdot c \circ I^n_{(i',\alpha')}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right) - 2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$$

Now, we will analyze the right side of the previous equation:

- Because $c^{(1,j,\beta)}$ is an odd edge then $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{c^{(1,j,\beta)}}\right)$ is an odd number.
- Because $\overline{c^{(1,j,\beta)}}$ is an even number then $Card\left([(-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}]^{R_P}_{\overline{c^{(1,j,\beta)}}}\right)$ is an even number.
- Obviously $2 \cdot Card\left(\eta^{c^{(1,j,\beta)}}\right)$ is an even number.

$\therefore Card\left(\wp^{c^{(1,j,\beta)}} \cup \wp^{\overline{c^{(1,j,\beta)}}}\right)$ is an odd number.

⇐)
The reciprocal is the counterreciprocal of **Lemma 4.3** (p ⇒ q ≡¬q ⇒ ¬p). ◪

**Corollary 4.2:** *Let c be a combination of nD hyper-boxes. In c exists a pair of collinear odd edges, or a pair of collinear even edges, both on axis $x_j$, $1 \le j \le n$, if and only if the total number of (n-1)D cells in $\partial(c)$ which are perpendicular to such edges is even.*
Proof:
The proposition is the counterreciprocal of **Theorem 4.4** (p ⇔ q ≡ ¬p ⇔ ¬q). ◪

**Definition 4.22:** *Let $c^{(1,j,\beta)}$ be a main edge and let $c_1, c_2, \ldots, c_k$ a combination of nD hyper-boxes with oriented cells $(-1)^{i+\alpha} \cdot c_1 \circ I^n_{(i,\alpha)}, \ldots, (-1)^{i+\alpha} \cdot c_k \circ I^n_{(i,\alpha)}$ respectively, $1 \le i \le n$, $\alpha \in \{0,1\}$. We define the relation $\underline{R_A^{c^{(1,j,\beta)}}}$ as:*

$$c_k \circ I^n_{(i,\alpha)} \; R_A^{c^{(1,j,\beta)}} \; c_{k'} \circ I^n_{(i',\alpha')} \Leftrightarrow ([c^{(1,j,\beta)} \text{ is adjacent to } c_k \circ I^n_{(i,\alpha)}] \wedge [c^{(1,j,\beta)} \text{ is adjacent to } c_{k'} \circ I^n_{(i',\alpha')}])$$

*where $1 \le k' \le k$, $1 \le i' \le n$, $\alpha' \in \{0,1\}$.*

**Theorem 4.5:** *Relation $R_A^{c^{(1,j,\beta)}}$ is an equivalence relation.*
Proof:

Let $(-1)^{i+\alpha} \cdot c_k \circ I^n_{(i,\alpha)}$, $(-1)^{i+\alpha} \cdot c_{k'} \circ I^n_{(i',\alpha')}$, $(-1)^{i+\alpha} \cdot c_{k''} \circ I^n_{(i'',\alpha'')}$ be oriented cells of general singular nD hyper-boxes, $1 \le i, i', i'' \le n$, $\alpha, \alpha', \alpha'' \in \{0,1\}$.

The following properties are satisfied:

- Reflexivity: $(\forall (-1)^{i+\alpha} \cdot c \circ I^n_{(i,\alpha)}, 1 \le i \le n, \alpha \in \{0,1\})(c \circ I^n_{(i,\alpha)} \; R_A^{c^{(1,j,\beta)}} \; c \circ I^n_{(i,\alpha)})$
- Symmetry:
  If $c_k \circ I^n_{(i,\alpha)} \; R_A^{c^{(1,j,\beta)}} \; c_{k'} \circ I^n_{(i',\alpha')}$
  $\Rightarrow ([c^{(1,j,\beta)}([0,1]) \subseteq c_k(I^n_{(i,\alpha)}([0,1]^n))] \wedge [c^{(1,j,\beta)}([0,1]) \subseteq c_{k'}(I^n_{(i',\alpha')}([0,1]^n))])$
  $\Rightarrow ([c^{(1,j,\beta)}([0,1]) \subseteq c_{k'}(I^n_{(i',\alpha')}([0,1]^n))] \wedge [c^{(1,j,\beta)}([0,1]) \subseteq c_k(I^n_{(i,\alpha)}([0,1]^n))])$
  $\Rightarrow c_{k'} \circ I^n_{(i',\alpha')} \; R_A^{c^{(1,j,\beta)}} \; c_k \circ I^n_{(i,\alpha)}$
  $\therefore (\forall (-1)^{i+\alpha} \cdot c_k \circ I^n_{(i,\alpha)}, \; 1 \le i \le n, \; \alpha \in \{0,1\})(\forall (-1)^{i+\alpha} \cdot c_{k'} \circ I^n_{(i',\alpha')}, \; 1 \le i' \le n, \; \alpha' \in \{0,1\})$
  $(c_k \circ I^n_{(i,\alpha)} \; R_A^{c^{(1,j,\beta)}} \; c_{k'} \circ I^n_{(i',\alpha')} \Rightarrow c_{k'} \circ I^n_{(i',\alpha')} \; R_A^{c^{(1,j,\beta)}} \; c_k \circ I^n_{(i,\alpha)})$

- Transitivity:

If $c_k \circ I^n_{(i,\alpha)}\ R_A^{c^{(1,j,\beta)}}\ c_{k'} \circ I^n_{(i',\alpha')} \wedge c_{k'} \circ I^n_{(i',\alpha')}\ R_A^{c^{(1,j,\beta)}}\ c_{k''} \circ I^n_{(i'',\alpha'')}$

$\Rightarrow ([c^{(1,j,\beta)}([0,1]) \subseteq c_k(I^n_{(i,\alpha)}([0,1]^n))] \wedge [c^{(1,j,\beta)}([0,1]) \subseteq c_{k'}(I^n_{(i',\alpha')}([0,1]^n))]) \wedge$

$([c^{(1,j,\beta)}([0,1]) \subseteq c_{k'}(I^n_{(i',\alpha')}([0,1]^n))] \wedge [c^{(1,j,\beta)}([0,1]) \subseteq c_{k''}(I^n_{(i'',\alpha'')}([0,1]^n))])$

$\Rightarrow ([c^{(1,j,\beta)}([0,1]) \subseteq c_k(I^n_{(i,\alpha)}([0,1]^n))] \wedge [c^{(1,j,\beta)}([0,1]) \subseteq c_{k''}(I^n_{(i'',\alpha'')}([0,1]^n))])$

$\Rightarrow c_k \circ I^n_{(i,\alpha)}\ R_A^{c^{(1,j,\beta)}}\ c_{k''} \circ I^n_{(i'',\alpha'')}$

$\therefore (\forall(-1)^{i+\alpha} \cdot c_k \circ I^n_{(i,\alpha)},\ 1 \le i \le n,\ \alpha \in \{0,1\})(\forall(-1)^{i+\alpha} \cdot c_{k'} \circ I^n_{(i',\alpha')},\ 1 \le i' \le n,\ \alpha' \in \{0,1\})$

$(\forall(-1)^{i+\alpha} \cdot c_{k''} \circ I^n_{(i'',\alpha'')},\ 1 \le i'' \le n,\ \alpha'' \in \{0,1\})$

$(c_k \circ I^n_{(i,\alpha)}\ R_A^{c^{(1,j,\beta)}}\ c_{k'} \circ I^n_{(i',\alpha')} \wedge c_{k'} \circ I^n_{(i',\alpha')}\ R_A^{c^{(1,j,\beta)}}\ c_{k''} \circ I^n_{(i'',\alpha'')} \Rightarrow c_k \circ I^n_{(i,\alpha)}\ R_A^{c^{(1,j,\beta)}}\ c_{k''} \circ I^n_{(i'',\alpha'')})$

$\therefore$ Relation $R_A^{c^{(1,j,\beta)}}$ is an equivalence relation. ◪

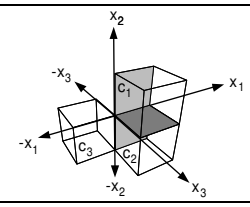
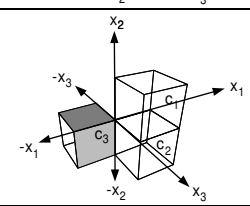
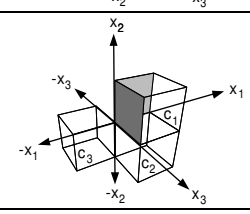
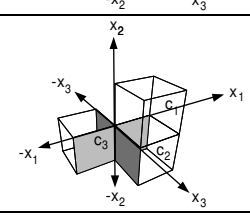
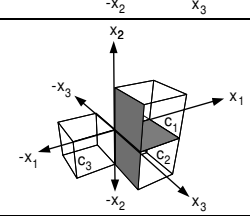
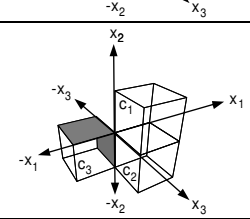
| Main Edge | Adjacent Cells | Equivalence Classes | |
|---|---|---|---|
| $c^{(1,1,0)}(x_1) = (x_1, 0, 0)$ | $c_1(I^3_{(2,0)}(x)) = (x_1, 0, x_2)$<br>$c_1(I^3_{(3,0)}(x)) = (x_1, x_2, 0)$<br>$c_2(I^3_{(3,0)}(x)) = (x_1, x_2 - 1, 0)$<br>$c_2(I^3_{(2,1)}(x)) = (x_1, 0, x_2)$ | $\left[1 \cdot c_1(I^3_{(2,0)}(x))\right]^{R_A}_{c^{(1,1,0)}} =$<br>$\{1 \cdot c_1(I^3_{(2,0)}(x)),\ -1 \cdot c_1(I^3_{(3,0)}(x)),$<br>$-1 \cdot c_2(I^3_{(3,0)}(x)),\ -1 \cdot c_2(I^3_{(2,1)}(x))\}$<br>($c_1$ and $c_2$ have common faces but with opposite orientations) |  |
| $c^{(1,1,1)}(x_1) = (-x_1, 0, 0)$ | $c_3(I^3_{(2,1)}(x)) = (x_1 - 1, 0, x_2 - 1)$<br>$c_3(I^3_{(3,1)}(x)) = (x_1 - 1, x_2 - 1, 0)$ | $\left[-1 \cdot c_3(I^3_{(2,1)}(x))\right]^{R_A}_{c^{(1,1,1)}} =$<br>$\{-1 \cdot c_3(I^3_{(2,1)}(x)),\ 1 \cdot c_3(I^3_{(3,1)}(x))\}$ |  |
| $c^{(1,2,0)}(x_1) = (0, x_1, 0)$ | $c_1(I^3_{(1,0)}(x)) = (0, x_1, x_2)$<br>$c_1(I^3_{(3,0)}(x)) = (x_1, x_2, 0)$ | $\left[-1 \cdot c_1(I^3_{(1,0)}(x))\right]^{R_A}_{c^{(1,2,0)}} =$<br>$\{-1 \cdot c_1(I^3_{(1,0)}(x)),\ -1 \cdot c_1(I^3_{(3,0)}(x))\}$ |  |
| $c^{(1,2,1)}(x_1) = (0, -x_1, 0)$ | $c_2(I^3_{(3,0)}(x)) = (x_1, x_2 - 1, 0)$<br>$c_2(I^3_{(1,0)}(x)) = (0, x_1 - 1, x_2)$<br>$c_3(I^3_{(3,1)}(x)) = (x_1 - 1, x_2 - 1, 0)$<br>$c_3(I^3_{(1,1)}(x)) = (0, x_1 - 1, x_2 - 1)$ | $\left[-1 \cdot c_2(I^3_{(3,0)}(x))\right]^{R_A}_{c^{(1,2,1)}} =$<br>$\{-1 \cdot c_2(I^3_{(3,0)}(x)),\ -1 \cdot c_2(I^3_{(1,0)}(x)),$<br>$1 \cdot c_3(I^3_{(3,1)}(x)),\ 1 \cdot c_3(I^3_{(1,1)}(x))\}$ |  |
| $c^{(1,3,0)}(x_1) = (0, 0, x_1)$ | $c_1(I^3_{(2,0)}(x)) = (x_1, 0, x_2)$<br>$c_1(I^3_{(1,0)}(x)) = (0, x_1, x_2)$<br>$c_2(I^3_{(2,1)}(x)) = (x_1, 0, x_2)$<br>$c_2(I^3_{(1,0)}(x)) = (0, x_1 - 1, x_2)$ | $\left[-1 \cdot c_1(I^3_{(3,0)}(x))\right]^{R_A}_{c^{(1,3,0)}} =$<br>$\{1 \cdot c_1(I^3_{(2,0)}(x)), -1 \cdot c_1(I^3_{(1,0)}(x)),$<br>$-1 \cdot c_2(I^3_{(2,1)}(x)),\ -1 \cdot c_2(I^3_{(1,0)}(x))\}$<br>($c_1$ and $c_2$ have common faces but with opposite orientations) |  |
| $c^{(1,3,1)}(x_1) = (0, 0, -x_1)$ | $c_3(I^3_{(1,1)}(x)) = (0, x_1 - 1, x_2 - 1)$<br>$c_3(I^3_{(2,1)}(x)) = (x_1 - 1, 0, x_2 - 1)$ | $\left[1 \cdot c_3(I^3_{(1,1)}(x))\right]^{R_A}_{c^{(1,3,1)}} =$<br>$\{1 \cdot c_3(I^3_{(1,1)}(x)),\ -1 \cdot c_3(I^3_{(2,1)}(x))\}$ |  |

**Table 4.5.** Computing the Equivalence Classes under relation $R_A$ of the 3D combination whose cells are described in **Table 4.2**.

**Definition 4.23:** *Consider equivalence relation* $R_A^{c^{(1,j,\beta)}}$. *The set*

$$\left[(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}\right]^{R_A}_{c^{(1,j,\beta)}}=\{(-1)^{i'+\alpha'}\cdot c_{k'}\circ I^n_{(i',\alpha')} : c_{k'}\circ I^n_{(i',\alpha')}\ R_A^{c^{(1,j,\beta)}}\ c_k\circ I^n_{(i,\alpha)}\}$$

*is the* <u>*equivalence class under*</u> $R_A^{c^{(1,j,\beta)}}$ <u>*of the oriented cell*</u> $(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}$ *induced by the main edge* $c^{(1,j,\beta)}$ *and whose representative is* $(-1)^{i+\alpha}\cdot c_k\circ I^n_{(i,\alpha)}$.

For example, consider the 3D combination of boxes whose 2D cells are described in **Table 4.2** (remember that these boxes and cells are defined in lattice $L^3_{(1,1,1)}$). Now, we identify the cells that are adjacent to each main edge in $\mathbb{R}^3$. Based in this information we build the equivalence classes induced by these edges and cells. See **Table 4.5**.

**Definition 4.24:** *Consider the oriented cells* $(-1)^{i+\alpha}\cdot c_k(I^n_{(i,\alpha)}(x))=(-1)^{i+\alpha}\cdot(x_1,x_2,...,x_n)$ *and* $(-1)^{i'+\alpha'}\cdot c_{k'}(I^n_{(i',\alpha')}(x'))=(-1)^{i'+\alpha'}\cdot(x'_1,x'_2,...,x'_n)$, $1\le i,i'\le n$, $\alpha,\alpha'\in\{0,1\}$. *We will say that* $c_k(I^n_{(i,\alpha)}(x))=(x_1,x_2,...,x_n)$ *and* $c_{k'}(I^n_{(i',\alpha')}(x'))=(x_1',x_2',...,x_n')$ *are* <u>*(n-1)D-coupled*</u> *if and only if:*

*a)* $(\forall i,1\le i\le n)(x_i=x'_i)$, *or well, if*

*b)* $(\exists!\, j,1\le j\le n)(\forall i\ne j)(x_i=x'_i)$

The above definition is in fact establishing that when we are referring to a pair of 1D-coupled oriented cells in $\mathbb{R}^2$ we denote collinear cells; if we are referring to a pair of 2D-coupled cells in $\mathbb{R}^3$ we denote coplanar cells, and so on.

**Definition 4.25:** *A set of various (n-1)D-coupled oriented cells will be called an* <u>*(n-1)D-couplet*</u>.

**Definition 4.26:** *Let* $\left[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}\right]^{R_A}_{c^{(1,j,\beta)}}$ *be an equivalence class induced by relation* $R_A^{c^{(1,j,\beta)}}$. *The partition of* $\left[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}\right]^{R_A}_{c^{(1,j,\beta)}}$ *is given by 2n-2 subsets* $P^{c^{(1,j,\beta)}}_{1^+}$, $P^{c^{(1,j,\beta)}}_{1^-}$, $P^{c^{(1,j,\beta)}}_{2^+}$, $P^{c^{(1,j,\beta)}}_{2^-}$, …, $P^{c^{(1,j,\beta)}}_{(n-1)^+}$, $P^{c^{(1,j,\beta)}}_{(n-1)^+}$ *such that two cells are in* $P^{c^{(1,j,\beta)}}_{k^+}$ *if and only if they are nD-coupled and they have positive orientation. In the other hand, two cells are in* $P^{c^{(1,j,\beta)}}_{k^+}$ *if and only if they are nD-coupled and they have negative orientation. Then*

$$[(-1)^{i+\alpha}\cdot c\circ I^n_{(i,\alpha)}]^{R_A}_{c^{(1,j,\beta)}}=\left[\bigcup_{k=1}^{n-1}P^{c^{(1,j,\beta)}}_{k^+}\right]\cup\left[\bigcup_{k=1}^{n-1}P^{c^{(1,j,\beta)}}_{k^-}\right]$$

Consider the 3D combination of boxes whose 2D cells are described in **Table 4.2**. We have the partitions presented in **Table 4.6**.

| Equivalence Class | Subsets $P^{c^{(1,j,\beta)}}_{1^+}$, $P^{c^{(1,j,\beta)}}_{1^-}$ | Subsets $P^{c^{(1,j,\beta)}}_{2^+}$, $P^{c^{(1,j,\beta)}}_{2^-}$ | Partition of the Equivalence Class |
|---|---|---|---|
| $\left[1\cdot c_1(I^3_{(2,0)}(x))\right]^{R_A}_{c^{(1,1,0)}}=$ $\{1\cdot c_1(I^3_{(2,0)}(x)),\ -1\cdot c_1(I^3_{(3,0)}(x)),$ $-1\cdot c_2(I^3_{(3,0)}(x)),\ -1\cdot c_2(I^3_{(2,1)}(x))\}$ | $P^{c^{(1,1,0)}}_{1^+}=\varnothing$ $P^{c^{(1,1,0)}}_{1^-}=\{-1\cdot c_1(I^3_{(3,0)}(x)),$ $-1\cdot c_2(I^3_{(3,0)}(x))\}$ | $P^{c^{(1,1,0)}}_{2^+}=\{1\cdot c_1(I^3_{(2,0)}(x))\}$ $P^{c^{(1,1,0)}}_{2^-}=\{-1\cdot c_2(I^3_{(2,1)}(x))\}$ | $\left[1\cdot c_1(I^3_{(2,0)}(x))\right]^{R_A}_{c^{(1,1,0)}}=$ $[P^{c^{(1,1,0)}}_{1^+}\cup P^{c^{(1,1,0)}}_{2^+}]\cup[P^{c^{(1,1,0)}}_{1^-}\cup P^{c^{(1,1,0)}}_{2^-}]$ |
| $\left[-1\cdot c_3(I^3_{(2,1)}(x))\right]^{R_A}_{c^{(1,1,1)}}=$ $\{-1\cdot c_3(I^3_{(2,1)}(x)),\ 1\cdot c_3(I^3_{(3,1)}(x))\}$ | $P^{c^{(1,1,1)}}_{1^+}=\varnothing$ $P^{c^{(1,1,1)}}_{1^-}=\{-1\cdot c_3(I^3_{(2,1)}(x))\}$ | $P^{c^{(1,1,1)}}_{2^+}=\{1\cdot c_3(I^3_{(3,1)}(x))\}$ $P^{c^{(1,1,1)}}_{2^-}=\varnothing$ | $\left[-1\cdot c_3(I^3_{(2,1)}(x))\right]^{R_A}_{c^{(1,1,1)}}=$ $[P^{c^{(1,1,1)}}_{1^+}\cup P^{c^{(1,1,1)}}_{2^+}]\cup[P^{c^{(1,1,1)}}_{1^-}\cup P^{c^{(1,1,1)}}_{2^-}]$ |
| $\left[-1\cdot c_1(I^3_{(1,0)}(x))\right]^{R_A}_{c^{(1,2,0)}}=$ $\{-1\cdot c_1(I^3_{(1,0)}(x)),\ -1\cdot c_1(I^3_{(3,0)}(x))\}$ | $P^{c^{(1,2,0)}}_{1^+}=\varnothing$ $P^{c^{(1,2,0)}}_{1^-}=\{-1\cdot c_1(I^3_{(1,0)}(x))\}$ | $P^{c^{(1,2,0)}}_{2^+}=\varnothing$ $P^{c^{(1,2,0)}}_{2^-}=\{-1\cdot c_1(I^3_{(3,0)}(x))\}$ | $\left[-1\cdot c_1(I^3_{(1,0)}(x))\right]^{R_A}_{c^{(1,2,0)}}=$ $[P^{c^{(1,2,0)}}_{1^+}\cup P^{c^{(1,2,0)}}_{2^+}]\cup[P^{c^{(1,2,0)}}_{1^-}\cup P^{c^{(1,2,0)}}_{2^-}]$ |
| $\left[-1\cdot c_2(I^3_{(3,0)}(x))\right]^{R_A}_{c^{(1,2,1)}}=$ $\{-1\cdot c_2(I^3_{(3,0)}(x)),\ -1\cdot c_2(I^3_{(1,0)}(x)),$ $1\cdot c_3(I^3_{(3,1)}(x)),\ 1\cdot c_3(I^3_{(1,1)}(x))\}$ | $P^{c^{(1,2,1)}}_{1^+}=\{1\cdot c_3(I^3_{(3,1)}(x))\}$ $P^{c^{(1,2,1)}}_{1^-}=\{-1\cdot c_2(I^3_{(3,0)}(x))\}$ | $P^{c^{(1,2,1)}}_{2^+}=\{1\cdot c_3(I^3_{(1,1)}(x))\}$ $P^{c^{(1,2,1)}}_{2^-}=\{-1\cdot c_2(I^3_{(1,0)}(x))\}$ | $\left[-1\cdot c_2(I^3_{(3,0)}(x))\right]^{R_A}_{c^{(1,2,1)}}=$ $[P^{c^{(1,2,1)}}_{1^+}\cup P^{c^{(1,2,1)}}_{2^+}]\cup[P^{c^{(1,2,1)}}_{1^-}\cup P^{c^{(1,2,1)}}_{2^-}]$ |
| $\left[-1\cdot c_1(I^3_{(3,0)}(x))\right]^{R_A}_{c^{(1,3,0)}}=$ $\{1\cdot c_1(I^3_{(2,0)}(x)),\ -1\cdot c_1(I^3_{(1,0)}(x)),$ $-1\cdot c_2(I^3_{(2,1)}(x)),\ -1\cdot c_2(I^3_{(1,0)}(x))\}$ | $P^{c^{(1,3,0)}}_{1^+}=\{1\cdot c_1(I^3_{(2,0)}(x))\}$ $P^{c^{(1,3,0)}}_{1^-}=\{-1\cdot c_2(I^3_{(2,1)}(x))\}$ | $P^{c^{(1,3,0)}}_{2^+}=\varnothing$ $P^{c^{(1,3,0)}}_{2^-}=\{-1\cdot c_1(I^3_{(1,0)}(x)),$ $-1\cdot c_2(I^3_{(1,0)}(x))\}$ | $\left[-1\cdot c_1(I^3_{(3,0)}(x))\right]^{R_A}_{c^{(1,3,0)}}=$ $[P^{c^{(1,3,0)}}_{1^+}\cup P^{c^{(1,3,0)}}_{2^+}]\cup[P^{c^{(1,3,0)}}_{1^-}\cup P^{c^{(1,3,0)}}_{2^-}]$ |
| $\left[1\cdot c_3(I^3_{(1,1)}(x))\right]^{R_A}_{c^{(1,3,1)}}=$ $\{1\cdot c_3(I^3_{(1,1)}(x)),-1\cdot c_3(I^3_{(2,1)}(x))\}$ | $P^{c^{(1,3,1)}}_{1^+}=\varnothing$ $P^{c^{(1,3,1)}}_{1^-}=\{-1\cdot c_3(I^3_{(2,1)}(x))\}$ | $P^{c^{(1,3,1)}}_{2^+}=\{1\cdot c_3(I^3_{(1,1)}(x))\}$ $P^{c^{(1,3,1)}}_{2^-}=\varnothing$ | $\left[1\cdot c_3(I^3_{(1,1)}(x))\right]^{R_A}_{c^{(1,3,1)}}=$ $[P^{c^{(1,3,1)}}_{1^+}\cup P^{c^{(1,3,1)}}_{2^+}]\cup[P^{c^{(1,3,1)}}_{1^-}\cup P^{c^{(1,3,1)}}_{2^-}]$ |

**Table 4.6.** Computing the partitions of the equivalence classes induced by relation $R_A^{c^{(1,j,\beta)}}$ in the 3D combination of cells described in **Table 4.2**.

**Definition 4.27:** *Let* $P_{k^+}^{c^{(1,j,\beta)}}$ *and* $P_{k^-}^{c^{(1,j,\beta)}}$, *1* $\leq k < n$, *be sets of oriented cells in the partition of equivalence class* $[(-1)^{i+\alpha}\cdot c\circ I_{(i,\alpha)}^{n}]_{c^{(1,j,\beta)}}^{R_A}$. *We define the sets* $\underline{\hbar_{k^+}^{c^{(1,j,\beta)}}}$ *and* $\underline{\hbar_{k^-}^{c^{(1,j,\beta)}}}$ *as follows:*

$$\hbar_{k^+}^{c^{(1,j,\beta)}} = \left\{ \begin{array}{l} (-1)^{i+\alpha}\cdot c\circ I_{(i,\alpha)}^{n} \in P_{k}^{c^{(1,j,\beta)}} : \\ (-1)^{i+\alpha}\cdot c\circ I_{(i,\alpha)}^{n} + (-1)^{i'+\alpha'}\cdot c\circ I_{(i',\alpha')}^{n} = 0,\ (-1)^{i'+\alpha'}\cdot c\circ I_{(i',\alpha')}^{n} \in P_{k^-}^{c^{(1,j,\beta)}} \end{array} \right\}$$

$$\hbar_{k^-}^{c^{(1,j,\beta)}} = \left\{ \begin{array}{l} (-1)^{i+\alpha}\cdot c\circ I_{(i,\alpha)}^{n} \in P_{k}^{c^{(1,j,\beta)}} : \\ (-1)^{i+\alpha}\cdot c\circ I_{(i,\alpha)}^{n} + (-1)^{i'+\alpha'}\cdot c\circ I_{(i',\alpha')}^{n} = 0,\ (-1)^{i'+\alpha'}\cdot c\circ I_{(i',\alpha')}^{n} \in P_{k^+}^{c^{(1,j,\beta)}} \end{array} \right\}$$

That is, the set $\hbar_{k^+}^{c^{(1,j,\beta)}}$ $\left(\hbar_{k^-}^{c^{(1,j,\beta)}}\right)$ contains the cells in $P_{k^+}^{c^{(1,j,\beta)}}$ $\left(P_{k^-}^{c^{(1,j,\beta)}}\right)$ that are also in $P_{k^-}^{c^{(1,j,\beta)}}$ $\left(P_{k^+}^{c^{(1,j,\beta)}}\right)$ but with opposite orientation. Such cells are not included in ∂(c).

Consider the partitions over the equivalence classes induced by relation $R_A^{c^{(1,j,\beta)}}$ which were shown in **Table 4.6**. The **Table 4.7** shows their corresponding sets $\hbar_{1^+}^{c^{(1,j,\beta)}}$, $\hbar_{1^-}^{c^{(1,j,\beta)}}$, $\hbar_{2^+}^{c^{(1,j,\beta)}}$ and $\hbar_{2^-}^{c^{(1,j,\beta)}}$.

| **Subsets** $P_{1^+}^{c^{(1,j,\beta)}}$**, and** $P_{1^-}^{c^{(1,j,\beta)}}$ | $\hbar_{1^+}^{c^{(1,j,\beta)}}$ **and** $\hbar_{1^-}^{c^{(1,j,\beta)}}$ | **Subsets** $P_{2^+}^{c^{(1,j,\beta)}}$**, and** $P_{2^-}^{c^{(1,j,\beta)}}$ | $\hbar_{2^+}^{c^{(1,j,\beta)}}$ **and** $\hbar_{2^-}^{c^{(1,j,\beta)}}$ |
|---|---|---|---|
| $P_{1^+}^{c^{(1,1,0)}} = \varnothing$<br>$P_{1^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,x_2,0),$<br>$-1\cdot(x_1,x_2-1,0)\}$ | $\hbar_{1^+}^{c^{(1,1,0)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,1,0)}} = \varnothing$ | $P_{2^+}^{c^{(1,1,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$P_{2^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $\hbar_{2^+}^{c^{(1,1,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$\hbar_{2^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,0,x_2)\}$ |
| $P_{1^+}^{c^{(1,1,1)}} = \varnothing$<br>$P_{1^-}^{c^{(1,1,1)}} = \{-1\cdot(x_1-1,0,x_2-1)\}$ | $\hbar_{1^+}^{c^{(1,1,1)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,1,1)}} = \varnothing$ | $P_{2^+}^{c^{(1,1,1)}} = \{1\cdot(x_1-1,x_2-1,0)\}$<br>$P_{2^-}^{c^{(1,1,1)}} = \varnothing$ | $\hbar_{2^+}^{c^{(1,1,1)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,1,1)}} = \varnothing$ |
| $P_{1^+}^{c^{(1,2,0)}} = \varnothing$<br>$P_{1^-}^{c^{(1,2,0)}} = \{-1\cdot(0,x_1,x_2)\}$ | $\hbar_{1^+}^{c^{(1,2,0)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,2,0)}} = \varnothing$ | $P_{2^+}^{c^{(1,2,0)}} = \varnothing$<br>$P_{2^-}^{c^{(1,2,0)}} = \{-1\cdot(x_1,x_2,0)\}$ | $\hbar_{2^+}^{c^{(1,2,0)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,2,0)}} = \varnothing$ |
| $P_{1^+}^{c^{(1,2,1)}} = \{1\cdot(x_1-1,x_2-1,0)\}$<br>$P_{1^-}^{c^{(1,2,1)}} = \{-1\cdot(x_1,x_2-1,0)\}$ | $\hbar_{1^+}^{c^{(1,2,1)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,2,1)}} = \varnothing$ | $P_{2^+}^{c^{(1,2,1)}} = \{1\cdot(0,x_1-1,x_2-1)\}$<br>$P_{2^-}^{c^{(1,2,1)}} = \{-1\cdot(0,x_1-1,x_2)\}$ | $\hbar_{2^+}^{c^{(1,2,1)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,2,1)}} = \varnothing$ |
| $P_{1^+}^{c^{(1,3,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$P_{1^-}^{c^{(1,3,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $\hbar_{1^+}^{c^{(1,3,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$\hbar_{1^-}^{c^{(1,3,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $P_{2^+}^{c^{(1,3,0)}} = \varnothing$<br>$P_{2^-}^{c^{(1,3,0)}} = \{-1\cdot(0,x_1,x_2),$<br>$-1\cdot(0,x_1-1,x_2)\}$ | $\hbar_{2^+}^{c^{(1,3,0)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,3,0)}} = \varnothing$ |
| $P_{1^+}^{c^{(1,3,1)}} = \varnothing$<br>$P_{1^-}^{c^{(1,3,1)}} = \{-1\cdot(x_1-1,0,x_2-1)\}$ | $\hbar_{1^+}^{c^{(1,3,1)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,3,1)}} = \varnothing$ | $P_{2^+}^{c^{(1,3,1)}} = \{1\cdot(0,x_1-1,x_2-1)\}$<br>$P_{2^-}^{c^{(1,3,1)}} = \varnothing$ | $\hbar_{2^+}^{c^{(1,3,1)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,3,1)}} = \varnothing$ |

**Table 4.7.** Computing the sets $\hbar_{1^+}^{c^{(1,j,\beta)}}$, $\hbar_{1^-}^{c^{(1,j,\beta)}}$, $\hbar_{2^+}^{c^{(1,j,\beta)}}$ and $\hbar_{2^-}^{c^{(1,j,\beta)}}$ for the 3D combination of cells described in **Table 4.2**.

**Definition 4.28:** *Let* $P_{k^+}^{c^{(1,j,\beta)}}$ *and* $P_{k^-}^{c^{(1,j,\beta)}}$, *1* $\leq k < n$, *be the sets of oriented cells that are included in the partition of the equivalence class* $[(-1)^{i+\alpha}\cdot c\circ I_{(i,\alpha)}^{n}]_{c^{(1,j,\beta)}}^{R_A}$. *We define to the sets* $\underline{\ell_{k^+}^{c^{(1,j,\beta)}}}$ *and* $\underline{\ell_{k^-}^{c^{(1,j,\beta)}}}$ *as follows:*

$$\ell_{k^+}^{c^{(1,j,\beta)}} = P_{k^+}^{c^{(1,j,\beta)}} \setminus \hbar_{k^+}^{c^{(1,j,\beta)}}$$
$$\ell_{k^-}^{c^{(1,j,\beta)}} = P_{k^-}^{c^{(1,j,\beta)}} \setminus \hbar_{k^-}^{c^{(1,j,\beta)}}$$

As can be seen, set $\ell_{k^+}^{c^{(1,j,\beta)}}$ $\left(\ell_{k^-}^{c^{(1,j,\beta)}}\right)$ contains to all the cells in $P_{k^+}^{c^{(1,j,\beta)}}$ $\left(P_{k^-}^{c^{(1,j,\beta)}}\right)$ except those which are included, with opposite orientation, in $P_{k^-}^{c^{(1,j,\beta)}}$ $\left(P_{k^+}^{c^{(1,j,\beta)}}\right)$. Consider for example the 3D combination, under lattice $L_{(1,1,1)}^3$, whose cells are described in **Table 4.2**. For each one of the sets that compose the partition of its equivalence classes we have their corresponding sets $\ell_{1^+}^{c^{(1,j,\beta)}}$ and $\ell_{1^-}^{c^{(1,j,\beta)}}$ which are described in **Table 4.8**, while **Table 4.9** shows its corresponding sets $\ell_{2^-}^{c^{(1,j,\beta)}}$ and $\ell_{2^+}^{c^{(1,j,\beta)}}$.

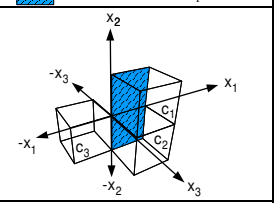
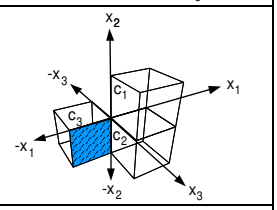
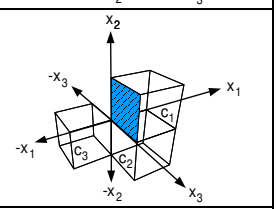
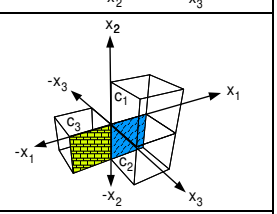
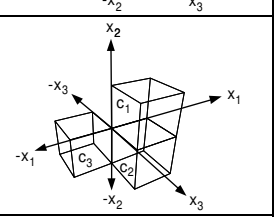
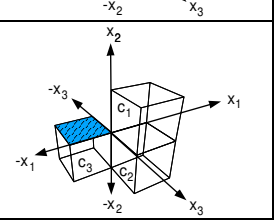
| **Sets** $P_{1^+}^{c^{(1,j,\beta)}}$ **and** $P_{1^-}^{c^{(1,j,\beta)}}$ | **Sets** $\hbar_{1^+}^{c^{(1,j,\beta)}}$ **and** $\hbar_{1^-}^{c^{(1,j,\beta)}}$ | $\ell_{1^+}^{c^{(1,j,\beta)}} = P_{1^+}^{c^{(1,j,\beta)}} \setminus \hbar_{1^+}^{c^{(1,j,\beta)}}$ **and** $\ell_{1^-}^{c^{(1,j,\beta)}} = P_{1^-}^{c^{(1,j,\beta)}} \setminus \hbar_{1^-}^{c^{(1,j,\beta)}}$ | **2D cells in** $\ell_{1^+}^{c^{(1,j,\beta)}}$ **2D cells in** $\ell_{1^-}^{c^{(1,j,\beta)}}$ |
|---|---|---|---|
| $P_{1^+}^{c^{(1,1,0)}} = \varnothing$<br>$P_{1^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,x_2,0),$<br>$-1\cdot(x_1,x_2-1,0)\}$ | $\hbar_{1^+}^{c^{(1,1,0)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,1,0)}} = \varnothing$ | $\ell_{1^+}^{c^{(1,1,0)}} = \varnothing \setminus \varnothing = \varnothing$<br>$\ell_{1^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,x_2,0),-1\cdot(x_1,x_2-1,0)\} \setminus \varnothing$<br>$= \{-1\cdot(x_1,x_2,0),-1\cdot(x_1,x_2-1,0)\}$ |  |
| $P_{1^+}^{c^{(1,1,1)}} = \varnothing$<br>$P_{1^-}^{c^{(1,1,1)}} = \{-1\cdot(x_1-1,0,x_2-1)\}$ | $\hbar_{1^+}^{c^{(1,1,1)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,1,1)}} = \varnothing$ | $\ell_{1^+}^{c^{(1,1,1)}} = \varnothing \setminus \varnothing = \varnothing$<br>$\ell_{1^-}^{c^{(1,1,1)}} = \{-1\cdot(x_1-1,0,x_2-1)\} \setminus \varnothing$<br>$= \{-1\cdot(x_1-1,0,x_2-1)\}$ |  |
| $P_{1^+}^{c^{(1,2,0)}} = \varnothing$<br>$P_{1^-}^{c^{(1,2,0)}} = \{-1\cdot(0,x_1,x_2)\}$ | $\hbar_{1^+}^{c^{(1,2,0)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,2,0)}} = \varnothing$ | $\ell_{1^+}^{c^{(1,2,0)}} = \varnothing \setminus \varnothing = \varnothing$<br>$\ell_{1^-}^{c^{(1,2,0)}} = \{-1\cdot(0,x_1,x_2)\} \setminus \varnothing$<br>$= \{-1\cdot(0,x_1,x_2)\}$ |  |
| $P_{1^+}^{c^{(1,2,1)}} = \{1\cdot(x_1-1,x_2-1,0)\}$<br>$P_{1^-}^{c^{(1,2,1)}} = \{-1\cdot(x_1,x_2-1,0)\}$ | $\hbar_{1^+}^{c^{(1,2,1)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,2,1)}} = \varnothing$ | $\ell_{1^+}^{c^{(1,2,1)}} = \{1\cdot(x_1-1,x_2-1,0)\} \setminus \varnothing$<br>$= \{1\cdot(x_1-1,x_2-1,0)\}$<br>$\ell_{1^-}^{c^{(1,2,1)}} = \{-1\cdot(x_1,x_2-1,0)\} \setminus \varnothing$<br>$= \{-1\cdot(x_1,x_2-1,0)\}$ |  |
| $P_{1^+}^{c^{(1,3,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$P_{1^-}^{c^{(1,3,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $\hbar_{1^+}^{c^{(1,3,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$\hbar_{1^-}^{c^{(1,3,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $\ell_{1^+}^{c^{(1,3,0)}} = \{1\cdot(x_1,0,x_2)\} \setminus \{1\cdot(x_1,0,x_2)\}$<br>$= \varnothing$<br>$\ell_{1^-}^{c^{(1,3,0)}} = \{-1\cdot(x_1,0,x_2)\} \setminus \{-1\cdot(x_1,0,x_2)\}$<br>$= \varnothing$ |  |
| $P_{1^+}^{c^{(1,3,1)}} = \varnothing$<br>$P_{1^-}^{c^{(1,3,1)}} = \{-1\cdot(x_1-1,0,x_2-1)\}$ | $\hbar_{1^+}^{c^{(1,3,1)}} = \varnothing$<br>$\hbar_{1^-}^{c^{(1,3,1)}} = \varnothing$ | $\ell_{1^+}^{c^{(1,3,1)}} = \varnothing \setminus \varnothing = \varnothing$<br>$\ell_{1^-}^{c^{(1,3,1)}} = \{-1\cdot(x_1-1,0,x_2-1)\} \setminus \varnothing$<br>$= \{-1\cdot(x_1-1,0,x_2-1)\}$ |  |

**Table 4.8.** Computing the sets $\ell_{1^+}^{c^{(1,j,\beta)}}$ and $\ell_{1^-}^{c^{(1,j,\beta)}}$ for the 3D combination of cells described in **Table 4.2**.

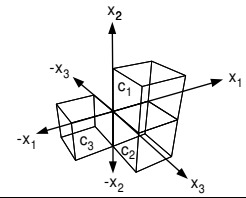
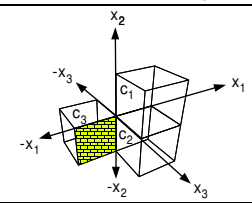
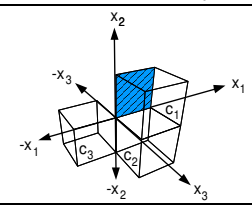
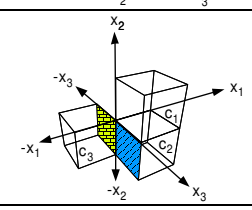
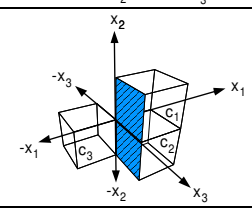
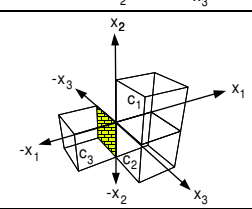
| **Sets** $P_{2^+}^{c^{(1,j,\beta)}}$ **and** $P_{2^-}^{c^{(1,j,\beta)}}$ | **Sets** $\hbar_{2^+}^{c^{(1,j,\beta)}}$ **and** $\hbar_{2^-}^{c^{(1,j,\beta)}}$ | $\ell_{2^+}^{c^{(1,j,\beta)}} = P_{2^+}^{c^{(1,j,\beta)}} \setminus \hbar_{2^+}^{c^{(1,j,\beta)}}$ **and** $\ell_{2^-}^{c^{(1,j,\beta)}} = P_{2^-}^{c^{(1,j,\beta)}} \setminus \hbar_{2^-}^{c^{(1,j,\beta)}}$ | **2D cells in** $\ell_{2^+}^{c^{(1,j,\beta)}}$ **2D cells in** $\ell_{2^-}^{c^{(1,j,\beta)}}$ |
|---|---|---|---|
| $P_{2^+}^{c^{(1,1,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$P_{2^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $\hbar_{2^+}^{c^{(1,1,0)}} = \{1\cdot(x_1,0,x_2)\}$<br>$\hbar_{2^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,0,x_2)\}$ | $\ell_{2^+}^{c^{(1,1,0)}} = \{1\cdot(x_1,0,x_2)\}\setminus\{1\cdot(x_1,0,x_2)\} = \varnothing$<br>$\ell_{2^-}^{c^{(1,1,0)}} = \{-1\cdot(x_1,0,x_2)\}\setminus\{-1\cdot(x_1,0,x_2)\} = \varnothing$ |  |
| $P_{2^+}^{c^{(1,1,1)}} = \{1\cdot(x_1-1,x_2-1,0)\}$<br>$P_{2^-}^{c^{(1,1,1)}} = \varnothing$ | $\hbar_{2^+}^{c^{(1,1,1)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,1,1)}} = \varnothing$ | $\ell_{2^+}^{c^{(1,1,1)}} = \{1\cdot(x_1-1,x_2-1,0)\}\setminus\varnothing = \{1\cdot(x_1-1,x_2-1,0)\}$<br>$\ell_{2^-}^{c^{(1,1,1)}} = \varnothing\setminus\varnothing = \varnothing$ |  |
| $P_{2^+}^{c^{(1,2,0)}} = \varnothing$<br>$P_{2^-}^{c^{(1,2,0)}} = \{-1\cdot(x_1,x_2,0)\}$ | $\hbar_{2^+}^{c^{(1,2,0)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,2,0)}} = \varnothing$ | $\ell_{2^+}^{c^{(1,2,0)}} = \varnothing\setminus\varnothing = \varnothing$<br>$\ell_{2^-}^{c^{(1,2,0)}} = \{-1\cdot(x_1,x_2,0)\}\setminus\varnothing = \{-1\cdot(x_1,x_2,0)\}$ |  |
| $P_{2^+}^{c^{(1,2,1)}} = \{1\cdot(0,x_1-1,x_2-1)\}$<br>$P_{2^-}^{c^{(1,2,1)}} = \{-1\cdot(0,x_1-1,x_2)\}$ | $\hbar_{2^+}^{c^{(1,2,1)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,2,1)}} = \varnothing$ | $\ell_{2^+}^{c^{(1,2,1)}} = \{1\cdot(0,x_1-1,x_2-1)\}\setminus\varnothing = \{1\cdot(0,x_1-1,x_2-1)\}$<br>$\ell_{2^-}^{c^{(1,2,1)}} = \{-1\cdot(0,x_1-1,x_2)\}\setminus\varnothing = \{-1\cdot(0,x_1-1,x_2)\}$ |  |
| $P_{2^+}^{c^{(1,3,0)}} = \varnothing$<br>$P_{2^-}^{c^{(1,3,0)}} = \{-1\cdot(0,x_1,x_2),$<br>$-1\cdot(0,x_1-1,x_2)\}$ | $\hbar_{2^+}^{c^{(1,3,0)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,3,0)}} = \varnothing$ | $\ell_{2^+}^{c^{(1,3,0)}} = \varnothing\setminus\varnothing = \varnothing$<br>$\ell_{2^-}^{c^{(1,3,0)}} = \{-1\cdot(0,x_1,x_2),-1\cdot(0,x_1-1,x_2)\}\setminus\varnothing = \{-1\cdot(0,x_1,x_2),-1\cdot(0,x_1-1,x_2)\}$ |  |
| $P_{2^+}^{c^{(1,3,1)}} = \{1\cdot(0,x_1-1,x_2-1)\}$<br>$P_{2^-}^{c^{(1,3,1)}} = \varnothing$ | $\hbar_{2^+}^{c^{(1,3,1)}} = \varnothing$<br>$\hbar_{2^-}^{c^{(1,3,1)}} = \varnothing$ | $\ell_{2^+}^{c^{(1,3,1)}} = \{1\cdot(0,x_1-1,x_2-1)\}\setminus\varnothing = \{1\cdot(0,x_1-1,x_2-1)\}$<br>$\ell_{2^-}^{c^{(1,3,1)}} = \varnothing\setminus\varnothing = \varnothing$ |  |

**Table 4.9.** Computing the sets $\ell_{2^+}^{c^{(1,j,\beta)}}$ and $\ell_{2^-}^{c^{(1,j,\beta)}}$ for the 3D combination of cells described in **Table 4.2**.

Although the following three propositions we will list are obvious (moreover, they are analogous to **Properties 4.1**, **4.2** and **4.3**), we present them as Properties because they will be very useful in proving some next results. It is easy to see that the sets $\ell_{k^+}^{c^{(1,j,\beta)}}$ and $\ell_{k^-}^{c^{(1,j,\beta)}}$ do not have common cells (even ignoring orientations), hence

**Property 4.4:** *The sets* $\ell_{k^+}^{c^{(1,j,\beta)}}$ *and* $\ell_{k^-}^{c^{(1,j,\beta)}}$ *are disjoint sets, i.e.,* $\ell_{k^+}^{c^{(1,j,\beta)}} \cap \ell_{k^-}^{c^{(1,j,\beta)}} = \varnothing$.

By ignoring orientations we have that the sets $\hbar_{k^+}^{c^{(1,j,\beta)}}$ and $\hbar_{k^-}^{c^{(1,j,\beta)}}$ have exactly the same (n-1)D cells. Therefore

**Property 4.5:** $Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right) = Card\left(\hbar_{k^-}^{c^{(1,j,\beta)}}\right)$

Finally, from the definition of $\ell_{k^+}^{c^{(1,j,\beta)}}$ we have that $\hbar_{k^+}^{c^{(1,j,\beta)}} \subseteq [(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^n]_{c^{(1,j,\beta)}}^{R_A}$; and from the definition of $\ell_{k^-}^{c^{(1,j,\beta)}}$ we have that $\hbar_{k^-}^{c^{(1,j,\beta)}} \subseteq [(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^n]_{c^{(1,j,\beta)}}^{R_A}$. Hence

**Property 4.6:** $Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}} \cap [(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^n]_{c^{(1,j,\beta)}}^{R_A}\right) = Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$ *and* $Card\left(\hbar_{k^-}^{c^{(1,j,\beta)}} \cap [(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^n]_{c^{(1,j,\beta)}}^{R_A}\right) = Card\left(\hbar_{k^-}^{c^{(1,j,\beta)}}\right)$

**Lemma 4.5:** *Consider a combination of nD hyper-boxes c. Let $c^{(1,j,\beta)}$ be a main edge on axis $x_j$, $1 \le j \le n$. Then, the total number of cells in the (n-1)D-couplet in ∂(c) which are incident to $c^{(1,j,\beta)}$ is given by*

$$Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - 2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$$

Proof:
Consider the following sets associated to $c^{(1,j,\beta)}$:

- $[(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^n]_{c^{(1,j,\beta)}}^{R_A}$
- $\ell_{k^+}^{c^{(1,j,\beta)}}$ and $\ell_{k^-}^{c^{(1,j,\beta)}}$
- $P_{k^+}^{c^{(1,j,\beta)}}$ and $P_{k^-}^{c^{(1,j,\beta)}}$
- $\hbar_{k^+}^{c^{(1,j,\beta)}}$ and $\hbar_{k^-}^{c^{(1,j,\beta)}}$

The set of cells in the nD-couplet in ∂(c) which are incident to $c^{(1,j,\beta)}$ is given by $\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}$. We will show that

$$Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right) = Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - 2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$$

$$Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right)$$
$$= Card\left(\ell_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(\ell_{k^-}^{c^{(1,j,\beta)}}\right) - Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cap \ell_{k^-}^{c^{(1,j,\beta)}}\right)$$

Because $\ell_{k^+}^{c^{(1,j,\beta)}}$ and $\ell_{k^-}^{c^{(1,j,\beta)}}$ are disjoint sets (**Property 4.4**):

$$= Card\left(\ell_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(\ell_{k^-}^{c^{(1,j,\beta)}}\right) - 0$$

By definition of $\ell_{k^+}^{c^{(1,j,\beta)}}$ and $\ell_{k^-}^{c^{(1,j,\beta)}}$ we have that:

$$= Card\left(P_{k^+}^{c^{(1,j,\beta)}} \setminus \hbar_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}} \setminus \hbar_{k^-}^{c^{(1,j,\beta)}}\right)$$

By a well known result in Set Theory (Card(A \ B) = Card(A) – Card(A ∩ B)):

$$= \left[Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) - Card\left(P_{k^+}^{c^{(1,j,\beta)}} \cap \hbar_{k^+}^{c^{(1,j,\beta)}}\right)\right] + \left[Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - Card\left(P_{k^-}^{c^{(1,j,\beta)}} \cap \hbar_{k^-}^{c^{(1,j,\beta)}}\right)\right]$$

By **Property 4.6**:

$$= \left[Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) - Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)\right] + \left[Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - Card\left(\hbar_{k^-}^{c^{(1,j,\beta)}}\right)\right]$$

Because **Property 4.5** states that $Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right) = Card\left(\hbar_{k^-}^{c^{(1,j,\beta)}}\right)$ then:

$$= Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - 2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$$

Finally we have:

$$Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right) = Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - 2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$$ ◪

**Lemma 4.6:** *Consider a combination of nD hyper-boxes c where there is an even $e_0$ edge on axis $x_j$, $1 \le j \le n$. Then, the total number of cells in the (n-1)D-couplet in ∂(c) which are incident to $e_0$ is even.*
Proof:
The even edge $e_0$ will be denoted by main edge $c^{(1,j,\beta)}$. Consider the following sets associated to $c^{(1,j,\beta)}$:

- $[(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^n]_{c^{(1,j,\beta)}}^{R_A}$
- $\ell_{k^+}^{c^{(1,j,\beta)}}$ and $\ell_{k^-}^{c^{(1,j,\beta)}}$
- $P_{k^+}^{c^{(1,j,\beta)}}$ and $P_{k^-}^{c^{(1,j,\beta)}}$
- $\hbar_{k^+}^{c^{(1,j,\beta)}}$ and $\hbar_{k^-}^{c^{(1,j,\beta)}}$

We will show that $Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right)$ is an even number.

By **Lemma 4.5** we have that

$$Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right) = Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - 2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$$

Because $P_{k^+}^{c^{(1,j,\beta)}}, P_{k^-}^{c^{(1,j,\beta)}}$ contain the (n-1)D-coupled cells incident to $c^{(1,j,\beta)}$ then each one of these cells belongs to only one nD hyper-box. By hypothesis $c^{(1,j,\beta)}$ is an even edge and therefore the number of nD hyper-boxes incident to $c^{(1,j,\beta)}$ is even. Hence, $Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right)$ is an even number.

$\therefore Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right)$ is an even number. ◪

**Theorem 4.6:** *Let c a combination of nD hyper-boxes. In combination c there is an odd $e_0$ edge on axis $x_j$, $1 \le j \le n$, if and only if the total number of cells in the (n-1)D-couplet in ∂(c) which are incident to $e_0$ is odd.*
Proof:
⇒)
Let the odd edge $e_0$ be denoted by main edge $c^{(1,j,\beta)}$. Consider the following sets associated to $c^{(1,j,\beta)}$:

- $[(-1)^{i+\alpha} \cdot c \circ I_{(i,\alpha)}^{n}]_{c^{(1,j,\beta)}}^{R_A}$
- $\ell_{k^+}^{c^{(1,j,\beta)}}$ and $\ell_{k^-}^{c^{(1,j,\beta)}}$
- $P_{k^+}^{c^{(1,j,\beta)}}$ and $P_{k^-}^{c^{(1,j,\beta)}}$
- $\hbar_{k^+}^{c^{(1,j,\beta)}}$ and $\hbar_{k^-}^{c^{(1,j,\beta)}}$

We will show that $Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right)$ is an odd number.

By **Lemma 4.5** we have that

$$Card\left(\ell_{k^+}^{c^{(1,j,\beta)}} \cup \ell_{k^-}^{c^{(1,j,\beta)}}\right) = Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right) - 2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$$

Because $P_{k^+}^{c^{(1,j,\beta)}}, P_{k^-}^{c^{(1,j,\beta)}}$ contain the (n-1)D-coupled cells incident to $c^{(1,j,\beta)}$ then each one of these cells belongs to only one nD hyper-box. By hypothesis $c^{(1,j,\beta)}$ is an odd edge and therefore the number of nD hyper-boxes incident to $c^{(1,j,\beta)}$ is odd. Hence, $Card\left(P_{k^+}^{c^{(1,j,\beta)}}\right) + Card\left(P_{k^-}^{c^{(1,j,\beta)}}\right)$ is an odd number. Obviously $2 \cdot Card\left(\hbar_{k^+}^{c^{(1,j,\beta)}}\right)$ is an even number, however, it discards from our counting those pairs of (n-1)D cells that are included in $P_{k^+}^{c^{(1,j,\beta)}}$ and $P_{k^-}^{c^{(1,j,\beta)}}$ but with opposite orientations.
⇐)
The reciprocal is the counterreciprocal of **Lemma 4.6** (p ⇒ q ≡¬q ⇒ ¬p). ◪

**Corollary 4.3:** *Let c a combination of nD hyper-boxes. In combination c there is an even $e_0$ edge on axis $x_j$, $1 \le j \le n$, if and only if the total number of cells in the (n-1)D-couplet in ∂(c) which are incident to $e_0$ is even.*
Proof:
The proposition is the counterreciprocal of **Theorem 4.6** (p ⇔ q ≡ ¬p ⇔ ¬q). ◪

**Theorem 4.7:** *Let c be a combination of nD hyper-boxes. Each odd edge of c is incident to:*

- *An even number of (n-1)D cells in ∂(c) if n is odd, or*
- *An odd number of (n-1)D cells in ∂(c) if n is even.*

Proof:
By **Theorem 4.6**, an odd edge $e_0$ embedded in axis $x_i$, $1 \le i \le n$, is incident to an odd number of (n-1)D-coupled cells in ∂(c). In n-Dimensional space we have n main (n-1)D hyperplanes which pass through the origin. The odd edge $e_0$ is embedded in all of these hyperplanes except one such that whose intersection with $e_0$ is only the origin. Hence, if n is an even number then n-1 is an odd number and the total sum of (n-1)D cells in ∂(c) incident to $e_0$ and embedded in each one of the n-1 hyperplanes is an odd number. In the other hand, if n is an odd number then n-1 is an even number and the total sum of (n-1)D cells in ∂(c) incident to $e_0$ and embedded in each one of the n-1 hyperplanes is an even number. ◪

**Corollary 4.4:** *Let c be a combination of nD hyper-boxes. Each even edge of c is incident to an even number of (n-1)D cells in ∂(c).*

Proof:

By **Corollary 4.3** an even edge $e_0$ embedded in axis $x_i$, $1 \leq i \leq n$, is incident to an even number of (n-1)D-coupled cells in ∂(c). Therefore, the total sum of (n-1)D cells in ∂(c) incident to $e_0$ and embedded in each one of the n-1 main (n-1)D hyperplanes, which pass through the origin, is an even number. ◪

## 4.4. Conclusions

In this chapter we have defined some frameworks and equivalence relations in order to demonstrate some properties related with the Odd Edge Characterization and the way it interacts with its incident (n-1)D cells in a combination of nD hyper-boxes. In this sense, Spivak's k-chains have been fruitful in providing the referred frameworks. Spivak's k-chains have allowed us to select, in a unambiguously and formal way, which (n-1)D cells, included in the boundary of a combination of nD hyper-boxes, to consider in order to establish the properties of an Odd or an Even edge from the local point of view of the combinatorial topology in the nD-OPP's.

As observed in **Appendix D**, the concepts of Odd and Even edge were implicitly present in the 1D, 2D, 3D and 4D-OPP's. Because Manifold edges in the 1D, 2D and 3D-OPP's have an odd number of incident segments, rectangles and boxes respectively [Aguilera98]; while Extreme edges in the 4D-OPP's have an odd number of incident 4D hyper-boxes [Pérez-Aguila03b], then all of them can be characterized as Odd Edges. As can be seen, the Odd Edge characterization have provide us an uniform framework based only in the fact of the oddity of the number of incident nD hyper-boxes to a given edge. Moreover the propositions

- In combination c there is an odd $e_0$ edge on axis $x_j$, $1 \leq j \leq n$, if and only if the total number of cells in the (n-1)D-couplet in ∂(c) which are incident to $e_0$ is odd (**Theorem 4.6**)
- In a combination c of nD hyper-boxes each odd edge of c is incident to an even number of (n-1)D cells in ∂(c) if n is odd; or, an odd number of (n-1)D cells in ∂(c) if n is even (**Theorem 4.7**)

extend such oddity property relating (n-1)D cells, incident to an odd edge, in the boundary of a combination of hyper-boxes. This framework together with the identified properties will lead us to establish, in **Chapter 5**, the fundaments behind the Extreme Vertices Model in the n-Dimensional Space.